1 Matching: Theory, Definition and Issues

Let $y(x,\varepsilon)$ be at least continuous on $D \times I$ where Let $D = [0,1]$, $I = (0,r)$. The function $y(x,\varepsilon)$ should be viewed as a solution of the algebraic problem

$$f(x,y,\varepsilon) = 0 \tag{1}$$

or a boundary-value problem like

$$\varepsilon \frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x) = f(x,\varepsilon), \tag{2}$$

$$y(0,\varepsilon) = A \tag{3}$$

$$y(1,\varepsilon) = B \tag{4}$$

Now, suppose that there are $y_k(x)$ such that the outer expansion

$$y \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots \tag{5}$$

is uniformly valid on $[\bar{x}, 1]$ and $\bar{x} > 0$. This expression is valid for x fixed in the limit $\varepsilon \to 0^+$. Now define

$$Y(X,\epsilon) = y(x,\varepsilon) \quad , \quad X = \frac{x}{\varepsilon} \tag{6}$$

suppose there exist $Y_k(X)$ such that the inner expansion

$$Y \sim Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + \cdots \tag{7}$$

uniformly on $[0, \bar{X}]$, for some $\bar{X} < 1/r$. as $\varepsilon \to 0^+$. This expression is valid for X fixed in the limit $\varepsilon \to 0^+$. The fixed x (outer) and fixed X (inner) limiting processes are depicted in Figure 1.

For clarity we let $D_o(\bar{x})$ and $D_i(\bar{X})$ denote the regions of uniform validity for the outer and inner expansions, respectively:

$$D_o(\bar{x}) = \{(x,\varepsilon) : x \in [\bar{x}, 1]\ ,\ \ \varepsilon \in I\} \tag{8}$$

$$D_i(\bar{X}) = \{(x,\varepsilon) : x \in [0, \bar{X}\varepsilon]\ , \varepsilon \in I\} \tag{9}$$

Though, as defined, $D_o(\bar{x})$ and $D_i(\bar{X})$ depend on the (ε-independent) fixed values $\bar{x}$ and $\bar{X}$, these values will be seen to be irrelevant to the latter discussions regarding overlap regions and matching. Therefore we will denote these regions simply as D_o and D_i. Extension theorems are theorems which extend the region of uniformity of asymptotic statements like (5). One early (and relatively simple) theorem is due to Kaplan (1967)[1]:

Theorem 1. *Let $D = [0,1]$, $I = (0,r)$ and $y(x,\varepsilon)$ be continuous on $D \times I$. Also, let $y_0(x)$ be some continuous function on $(0,1]$ such that*

$$\lim_{\varepsilon \to 0^+} [y(x,\varepsilon) - y_0(x)] = 0 \tag{10}$$

uniformly on $[\bar{x}, 1]$, for every $\bar{x} > 0$. Then there exists a function $0 < \delta(\varepsilon) \ll 1$ such that

$$\lim_{\varepsilon \to 0^+} [y(x,\varepsilon) - y_0(x)] = 0 \tag{11}$$

uniformly on $[\delta(\varepsilon), 1]$.

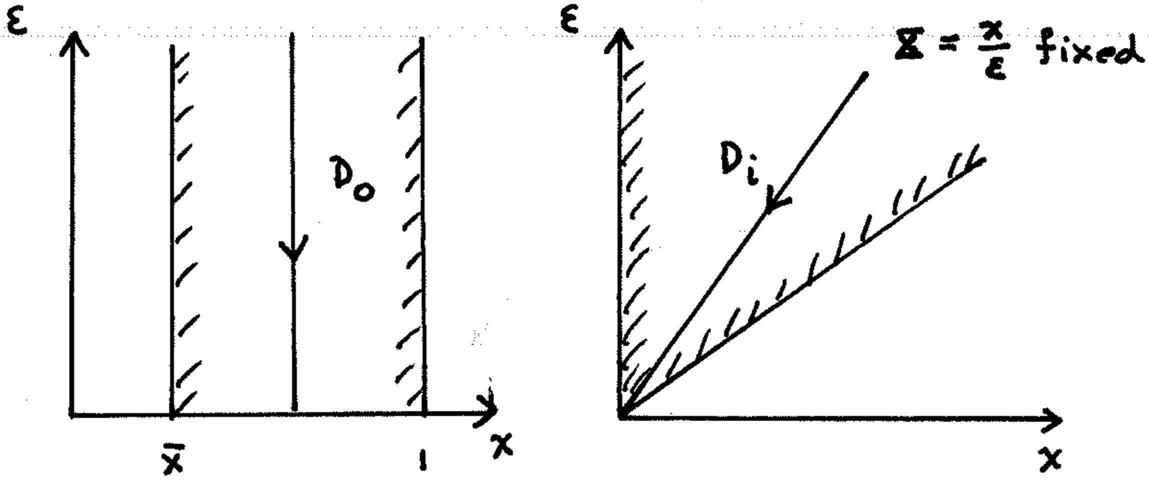

Figure 1: Depiction of outer (a) and inner (b) limiting processes

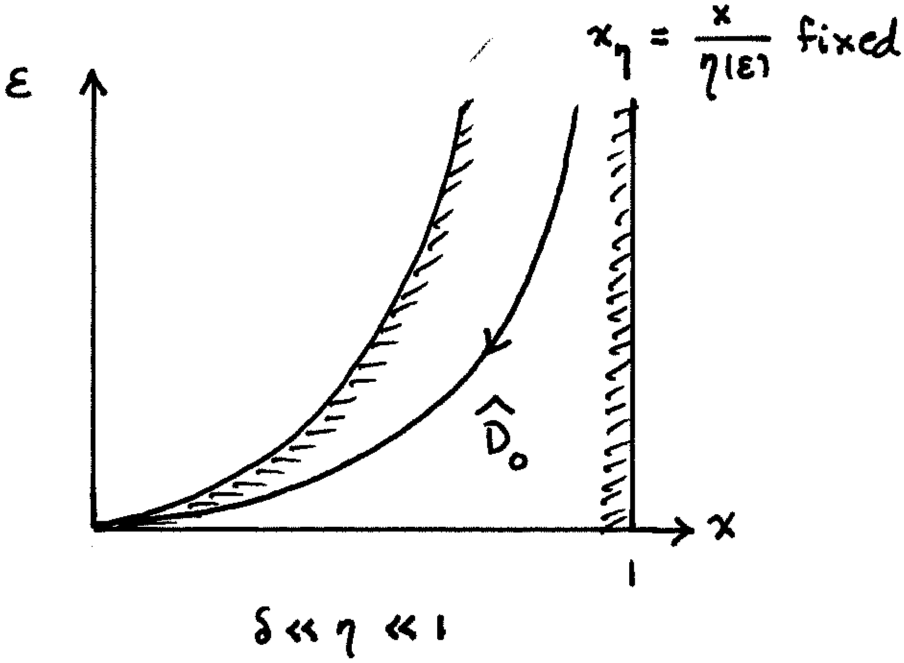

Figure 2: Extended domain of validity for leading outer expansion

A simple example of a function satisfying the hypothesis is:

$$y(x,\varepsilon) = x + e^{-x/\varepsilon} + \varepsilon \quad , \quad y_0(x) = x \tag{12}$$

since $y(x,\varepsilon) \sim y_0(x) + o(1)$ uniformly on $[\bar{x}, 1]$ and hence (10) is true.

What this theorem does is effectively extend the region of uniform validity D_o in Figure 1a to one like $\widehat{D}_o$ in Figure 2. To rigorously define $\widehat{D}_o$, intermediate variables need to be introduced. Let $\eta(\varepsilon)$ be any function with $0 < \eta(\varepsilon) \ll 1$. We define the intermediate variable x_η by

$$x = \eta(\varepsilon)x_\eta \tag{13}$$

Then, the conclusion of the theorem may be stated

$$\lim_{\varepsilon\to 0^+ x_\eta \; fixed} [y(\eta x_\eta, \varepsilon) - y_0(\eta x_\eta)] = 0 \tag{14}$$

uniformly on $x_\eta \in [\bar{x}_\eta, 1]$, for all η with $\delta = O(\eta)$. Generally, when introducing intermediate variables we view η as satisfying $\delta \ll \eta \ll 1$, though to clearly define $\widehat{D}_o$ we can set η equal to δ or 1:

$$\widehat{D}_o = \{(x,\varepsilon) : x \in [\bar{x}_\eta \delta(\varepsilon), 1], \varepsilon \in I\} \tag{15}$$

For the example in (12), we have for some intermediate variable x_η:

$$y(x,\varepsilon) - y_0(x) = e^{\frac{-x_\eta \eta}{\varepsilon}} + \varepsilon = o(1) \tag{16}$$

uniformly on $[\bar{x}_\eta, 1]$ providing $\bar{x}_\eta > 0$ and $\varepsilon \ll \eta$. For instance, one could choose $\delta(\varepsilon) = \varepsilon^{1/2}$ in the theorem.

In an analogous fashion, one can construct an extended domain of validity $\widehat{D}_i$ for the inner expansion (7) noting the inner variable

$$X = \frac{\eta x_\eta}{\varepsilon} \tag{17}$$

For some (x, ε) near $(0,0)$ the non-extended domains D_o and D_i do not *overlap* - see Figure 3a. Similarly, one can have nonoverlapping extended domains (Figure 3b) and overlapping extended domains (Figure 3c). If there is an overlapping extended domain, there are functions $\eta_i(\varepsilon)$ and $\eta_o(\varepsilon)$ such that for any intermediate variable x_η with $\eta_i(\varepsilon) \ll \eta(\varepsilon) \ll \eta_o(\varepsilon)$ both the inner and outer expansions are uniformly valid. That is to say, given any η with $\eta_i(\varepsilon) \ll \eta(\varepsilon) \ll \eta_o(\varepsilon)$, there is an ε-independent interval I_η such that both

$$\lim_{\varepsilon\to 0^+, x_\eta \; fixed} [y(\eta x_\eta, \varepsilon) - y_0(\eta x_\eta)] \;=\; 0 \tag{18}$$

$$\lim_{\varepsilon\to 0^+, x_\eta \; fixed} \left[y(\eta x_\eta, \varepsilon) - Y_0\left(\frac{\eta x_\eta}{\varepsilon}\right)\right] \;=\; 0 \tag{19}$$

uniformly on $x_\eta \in I_\eta$, $x_\eta > 0$. Subtracting these expressions we obtain a matching condition:

$$\lim_{\varepsilon\to 0^+, x_\eta \; fixed} \left[y_0(\eta x_\eta) - Y_0\left(\frac{\eta x_\eta}{\varepsilon}\right)\right] = 0 \tag{20}$$

And, if $y_0(0^+)$ and $Y_0(\infty)$ exist, since $\varepsilon \ll \eta \ll 1$,

$$\lim_{x\to 0^+} y_0(x) = \lim_{X\to\infty} Y_0(X) \tag{21}$$

[1] see Eckhaus (1979) for more theorems

which is called the *Prandtl matching condition.* If (20) can be satisfied, then one would say that the leading outer expansion $y_0(x)$ can be matched to the leading inner expansion $Y_0(X)$ on an overlap domain

$$\bar{D}_0 = \{(x,\varepsilon) : x_\eta = x\eta \in I_\eta, \eta_i(\varepsilon) \ll \eta(\varepsilon) \ll \eta_o(\varepsilon)\} \tag{22}$$

At this stage, we need to make a few points. Firstly, $y_0(0^+)$ or $Y_0(\infty)$ may not exist in which case the inner and outer expansions cannot be matched to leading-order using the Prandtl matching condition. However, it may still be possible to match the expansions by demonstrating the existence of an overlap domain for which (20) is satisfied. Secondly, even if the matching condition (20) cannot be satisfied that does not preclude the possibility of a P term outer expansion matching a Q term inner expansion. That is to say, there may be some overlap domain where

$$\lim_{\varepsilon\to 0^+, x_\eta \; fixed} \left[\sum_{n=0}^{P} \varepsilon^n y_n(x_\eta \eta) - \sum_{n=0}^{Q} \varepsilon^n Y_n\left(\frac{x_\eta \eta}{\varepsilon}\right)\right] = 0 \tag{23}$$

At this point we are in a position to define matching.

Definition 1. *Choose and fix $x_\eta = \frac{x}{\eta(\varepsilon)} \in \mathbb{R}$ and let R be any nonnegative integer. We say that the outer and inner expansions defined in (5)-(7) match to $O(\varepsilon^R)$ on a common domain of validity $\bar{D}_R(x_\eta)$ if there exist functions η_1 and η_2 with $\eta_1 \ll \eta_2$ and integers P, Q such that*

$$\lim_{\varepsilon\to 0^+, x_\eta \; fixed} \frac{M_{PQ}}{\varepsilon^R} = \lim_{\varepsilon\to 0^+, x_\eta \; fixed} \left[\frac{\sum_{n=0}^{P} \varepsilon^n y_n(x_\eta \eta) - \sum_{n=0}^{Q} \varepsilon^n Y_n\left(\frac{x_\eta \eta}{\varepsilon}\right)}{\varepsilon^R}\right] = 0 \tag{24}$$

for any function η satisfying $\eta_1 \ll \eta \ll \eta_2$ and

$$\bar{D}_R(x_\eta) = \{(x,\varepsilon) : x_\eta = x\eta, \eta_1(\varepsilon) \ll \eta(\varepsilon) \ll \eta_2(\varepsilon)\} \tag{25}$$

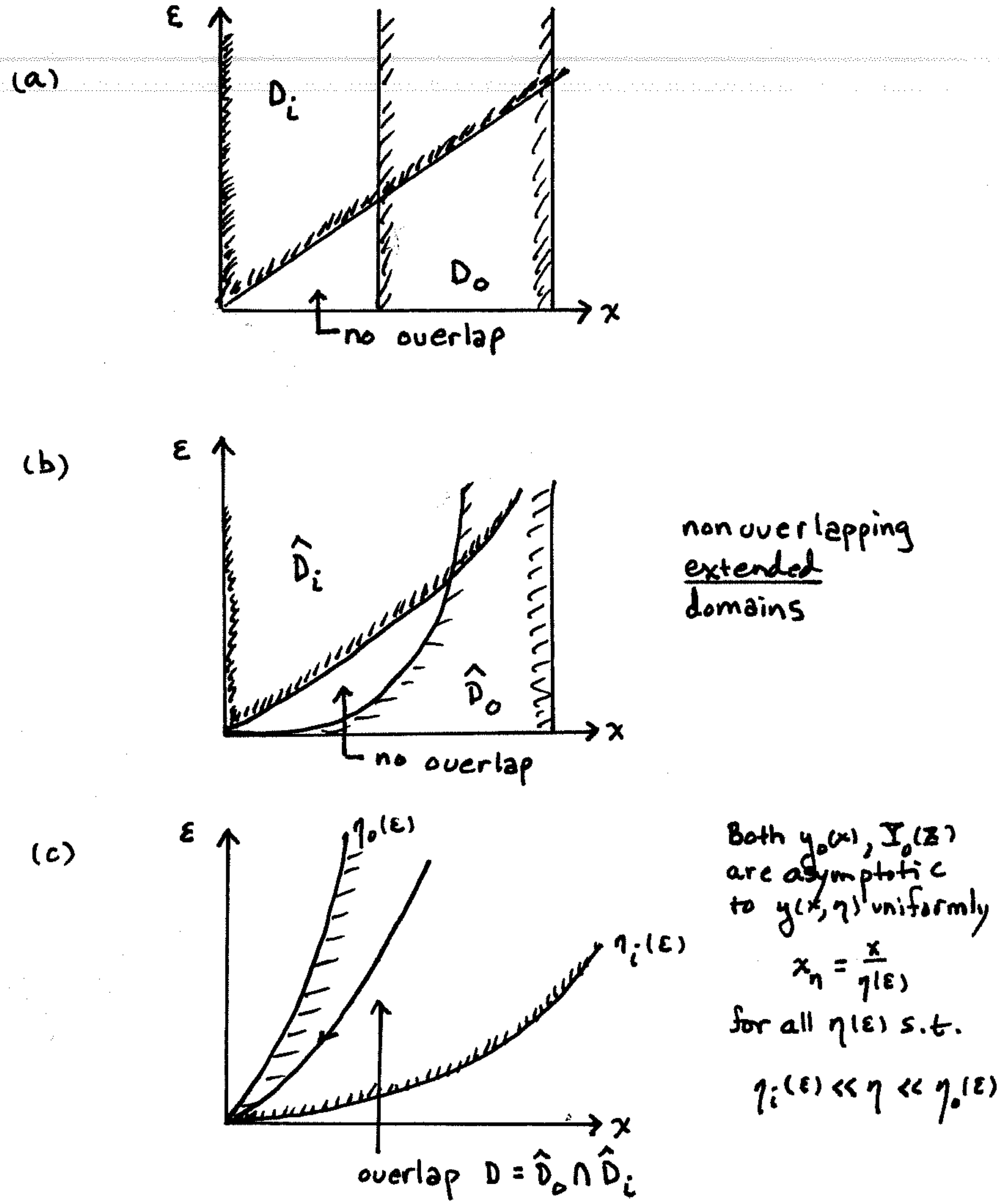

Figure 3: Depiction of (a) nonoverlapping nonextended domains, (b) nonoverlapping extended domains and (c) overlapping extended domains

We conclude with a few remarks:

1) General theorems showing the existence of overlap domains have not been found (Lagerstrom 1988). In practice, the existence of overlap domains where inner and outer solutions can be matched is done on a case by case basis.

2) For boundary value problems where the method of matched asymptotics is applied, matching conditions are used to find integration constants occurring in the inner expansion. Typically, inner and outer expansions can be matched only if those constants are chosen equal to specific values.

3) Prandtl matching corresponds to leading-order matching with $P = Q = R = 0$.

4) In some problems, P and Q may not be known apriori. Moreover, P may not equal Q.

5) Some expansions cannot be matched. The matching defined in (24) is with respect to the guage functions $\phi_n(\varepsilon) = \varepsilon^n, n \geq 0$. Clearly, some functions y may have more general outer expansions:

$$y(x,\varepsilon) \sim \sum_{n\geq 0} \phi_n(\varepsilon) y_n(x) \tag{26}$$

Indeed, the inner variable could be defined in a more general way, $X = x/\delta(\varepsilon), 0 < \delta \ll 1$, and the inner expansion may be with respect to different guage functions. These sorts of generalizations are not normally considered.

1.1 Model Problem

We consider a single example which illustrates all of the features discussed in the previous section. We will use the following facts throughout the discussion: For $x > 0$

$$(27)$$

$$|log(\varepsilon)| \ll \delta \;\Rightarrow\; e^{-\delta} \ll \varepsilon^n \quad \forall n > 0 \tag{28}$$

$$(29)$$

$$\delta = O_s(|log(\varepsilon)|) \;\Rightarrow\; e^{-\delta} = O_s(1) \tag{30}$$

$$(31)$$

$$x \ll \varepsilon|log(\varepsilon)| \;\Rightarrow\; e^{-x/\varepsilon} \ll \varepsilon^n \quad \forall n > 0 \tag{32}$$

$$(33)$$

where $\phi = O_s(\psi)$ means $\phi = O(\psi)$ and $\psi = O(\phi)$.

Specifically we will consider matching of inner and outer expansions of the function

$$y(t,\varepsilon) = \frac{1}{\sqrt{1-4\varepsilon}}\left\{exp\left[-(1-\sqrt{1-4\varepsilon})\frac{t}{2\varepsilon}\right] - exp\left[-(1+\sqrt{1-4\varepsilon})\frac{t}{2\varepsilon}\right]\right\} \tag{34}$$

which is the solution of the singular initial value problem

$$\varepsilon y'' + y' + y = 0 \quad , \tag{35}$$

$$y(0,\varepsilon) = 0 \quad , \quad y'(0,\varepsilon) = \frac{1}{\varepsilon} \tag{36}$$

The first two terms of the outer expansion

$$y(t,\varepsilon) \sim y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots \tag{37}$$

can easily be determined from (34). Fixing t and expanding in ε one finds

$$y = (1 + 2\varepsilon + O(\varepsilon^2)) \left[e^{-t-\varepsilon t+O(\varepsilon^2)} - \underbrace{e^{-t/\varepsilon + t + O(\varepsilon)}}_{T.S.T.} \right] \tag{38}$$

from which we deduce

$$y_0(t) = e^{-t} \quad , \quad y_1(t) = (2-t)e^{-t} \tag{39}$$

Similarly, to compute the inner expansion

$$y(t,\varepsilon) = Y(T,\varepsilon) \sim Y_0(T) + \varepsilon Y_1(T) + \varepsilon^2 Y_2(T) + \cdots \quad , \quad T = \frac{t}{\varepsilon} \tag{40}$$

reexpress (34) in terms of T, fix T and then expand in ε:

$$Y = (1 + 2\varepsilon + O(\varepsilon^2)) \left[e^{-\varepsilon T - \varepsilon^2 T + O(\varepsilon^3)} - e^{-T+\varepsilon T + O(\varepsilon^2)} \right] \tag{41}$$

From this one finds:

$$Y_0(T) = 1 - e^{-T} \quad , \quad Y_1(T) = (2-T) - (2+T)e^{-T} \tag{42}$$

Before we find the overlap domains where the outer and inner expansions match to $O(1)$ and $O(\varepsilon)$, we will discuss how these expansions would arise had we not known the exact solution *apriori*.

By substituting the expansion (37) into (35) we obtain the problems:

$$O(1) \ : \ y_0' + y_0 = 0 \tag{43}$$

$$O(\varepsilon) \ : \ y_1' + y_1 = -y_0'' \tag{44}$$

whose general solutions are (for a_0, b_0 constant)

$$y_0(t) = a_0 e^{-t} \tag{45}$$

$$y_1(t) = (b_0 - a_0 t)e^{-t} \tag{46}$$

Clearly, a_0 cannot be chosen so that $y_0(t)$ satisfy both initial conditions. Therefore, we surmise (postulate) that the true solution has layer at $t = 0$.

In terms of Y and T the initial value problem (35)-(36) can be written

$$Y'' + Y' + \varepsilon Y = 0 \quad , \tag{47}$$

$$Y(0,\varepsilon) = 0 \quad , \quad Y'(0,\varepsilon) = 1 \tag{48}$$

from which we obtain the inner problems

$$O(1) \; : \; Y_0'' + Y_0' = 0 \quad , \quad Y_0(0) = 0 \quad , \quad Y_0'(0) = 1 \tag{49}$$

$$O(\varepsilon) \; : \; Y_1'' + Y_1' = -Y_0 \quad , \quad Y_1(0) = 0 \quad , \quad Y_1'(0) = 0 \tag{50}$$

whose solutions are that given in (42). In contrast to boundary value problems, the unknown constants of integration to be determined from matching are part of the outer solution. If we apply Prandtl matching to match y_0 and Y_0 we find

$$\lim_{t \to 0^+} y_0(t) = a_0 = 1 = \lim_{T \to \infty} Y_0(T) \tag{51}$$

and recover $y_0(t)$ in (39).

Demonstrating extended outer domains to $O(1)$

To find an extended domain for the outer expansion one assumes $\eta(\varepsilon) \ll 1$ and seeks an $\eta_1(\varepsilon)$ such that $\eta_1(\varepsilon) \ll \eta(\varepsilon)$ implies

$$\lim_{\varepsilon \to 0^+, t_\eta \; fixed} [y(\eta t_\eta, \varepsilon) - y_0(\eta t_\eta)] = 0 \tag{52}$$

for the intermediate variable

$$t_\eta = \frac{t}{\eta} > 0 \tag{53}$$

Given (38), this limit holds providing $e^{-t_\eta \eta/\varepsilon} \ll 1$. To assure this, we choose $\eta_1(\varepsilon) = \varepsilon|log(\varepsilon)|$. Now let the notation $\phi \ll= \psi$ mean that either $\phi \ll \psi$ or $\phi = O_s(\psi)$. Then we can conclude that $\bar{D}_o$ will be an extended domain for the outer expansion so long as η satisfies

$$\eta_{1,0} \equiv \varepsilon|log(\varepsilon)| \ll \eta \ll= 1 \tag{54}$$

Though each η defines a different region in the (x,ε)-plane for $t_\eta \in I_\eta$, all that really matters for the limit to vanish is that η satisfy (54). So it is common practice to say that the extended domain for the single term outer expansion $y_0(t)$. "is" (54).

Demonstrating extended outer domains to $O(\varepsilon)$

To find the extended domain for the two term outer expansion $y_0(t) + \varepsilon y_1(t)$ one assumes $\eta(\varepsilon) \ll 1$ and seeks an $\eta_{1,1}(\varepsilon)$ such that $\eta_{1,1}(\varepsilon) \ll \eta(\varepsilon)$ implies

$$\lim_{\varepsilon \to 0^+, t_\eta \; fixed} \frac{[y(\eta t_\eta, \varepsilon) - y_0(\eta t_\eta) - \varepsilon y_1(\eta t_\eta)]}{\varepsilon} = 0 \tag{55}$$

Again from (38), we find that if η satisfies (54) the above limit holds. That is to say the choice $\eta_{1,1} = \eta_{1,0}$ works. If we continue this process of extending the domain in an R term outer expansion to find $\eta_{1,R}$ it is often the case that $\eta_{1,R} \ll \eta_{1,R+1}$ since adding more terms to the limit places more restrictions on η. For this particular example the extended outer domains at $O(1)$ and $O(\varepsilon)$ turned out to be the same.

Demonstrating extended inner domains to $O(1)$ and $O(\varepsilon)$
To find an extended domain for the single term inner expansion one assumes $\varepsilon \ll \eta(\varepsilon)$ and seeks an $\eta_2(\varepsilon)$ such that $\eta \ll \eta_2(\varepsilon)$ implies

$$\lim_{\varepsilon\to 0^+, t_\eta \text{ fixed}} [y(\eta t_\eta, \varepsilon) - Y_0(\eta t_\eta/\varepsilon)] = 0 \tag{56}$$

Again from (38) it is easy to verify that the extended domain for the single term inner expansion is defined by

$$\varepsilon \lll= \eta \ll \eta_{2,0} \equiv 1 \tag{57}$$

Finding the extended inner domain to $O(\varepsilon)$ is more delicate. In terms of the intermediate variables

$$\frac{y(\eta t_\eta, \varepsilon)}{\varepsilon} = \frac{1}{\varepsilon} - \frac{e^{-t_\eta \eta/\varepsilon}}{\varepsilon} - \frac{\eta}{\varepsilon} t_\eta - \frac{\eta}{\varepsilon} t_\eta e^{-t_\eta \eta/\varepsilon} + 2 - 2e^{-t_\eta \eta/\varepsilon} + O(\eta) + O\left(\frac{\eta^2}{\varepsilon}\right) + O(\varepsilon) \tag{58}$$

and in terms of the intermediate variables

$$\frac{1}{\varepsilon} Y_0 + Y_1 = \frac{y(\eta t_\eta, \varepsilon)}{\varepsilon} = \frac{1}{\varepsilon} - \frac{e^{-t_\eta \eta/\varepsilon}}{\varepsilon} - \frac{\eta}{\varepsilon} t_\eta - \frac{\eta}{\varepsilon} t_\eta e^{-t_\eta \eta/\varepsilon} + 2 - 2e^{-t_\eta \eta/\varepsilon} \tag{59}$$

Subtracting these two expressions we see that

$$\lim_{\varepsilon\to 0^+, t_\eta \text{ fixed}} \frac{[y(\eta t_\eta, \varepsilon) - Y_0(\eta t_\eta/\varepsilon) - \varepsilon Y_1(\eta t_\eta/\varepsilon)]}{\varepsilon} = 0 \tag{60}$$

provided $\eta^2/\varepsilon \ll 1$. That is to say the choice $\eta_{2,1} = \varepsilon^{1/2}$ ensures the limit vanishes and the extended inner domain to $O(\varepsilon)$ is

$$\varepsilon \lll= \eta \ll \eta_{2,1} \equiv \varepsilon^{1/2} \tag{61}$$

Here we note the extended domain to $O(\varepsilon)$ is "smaller" than the domain to $O(1)$, i.e. $\eta_{2,1} \ll \eta_{2,0}$.

Demonstrating overlap to $O(1)$ and $O(\varepsilon)$
Considering the previous discussions it is clear to see that the overlap domains to $O(1)$ and $O(\varepsilon)$ are, respectively,

$$\eta_{1,0} \ll \eta \ll \eta_{2,0} \tag{62}$$
$$\eta_{1,1} \ll \eta \ll \eta_{2,1} \tag{63}$$

or

$$\varepsilon |log(\varepsilon)| \ll \eta \ll 1 \tag{64}$$
$$\varepsilon |log(\varepsilon)| \ll \eta \ll \varepsilon^{1/2} \tag{65}$$

If η satisfies these asymptotic relations, the outer and inner expansions match to $O(1)$ and $O(\varepsilon)$, respectively. Explicitly, if η satisfies (62) then

$$\lim_{\varepsilon\to 0^+, t_\eta \text{ fixed}} [y_0(\eta t_\eta) - Y_0(\eta t_\eta/\varepsilon)] = 0 \tag{66}$$

And, if η satisfies the more stringent requirement (63)

$$\lim_{\varepsilon\to 0^+, t_\eta \ fixed} \frac{[y_0(\eta t_\eta) + \varepsilon y_1(\eta t_\eta) - Y_0(\eta t_\eta/\varepsilon) - \varepsilon Y_1(\eta t_\eta/\varepsilon)]}{\varepsilon} = 0 \tag{67}$$

If the exact solution y was not known *apriori* then one would choose a_0 in the incomplete outer solution $y_0(t) = a_0 e^{-t}$ and find $\eta_{1,0}, \eta_{2,0}$ so that (66) is satisfied.