Convex Functionals

Let X be a linear space and $J: A \to \mathbb{R}$ where $A \subset X$ is some admissible set.
We assume $\delta J(u; v)$ exists for all $u \in A$ and $v \in A^*$

Defn: $J: A \subset X \to \mathbb{R}$ is convex on A if

$$J(u+v) - J(u) \geq \delta J(u; v)$$

for all $u \in A$, $v \in A^*$. J is strictly convex iff equality holds for $v \equiv 0$.

Remark: This defn is specific to optimization theory and less general than that for metric spaces (for instance)

Theorem: If J is [strictly] convex on A and $\exists\, \bar{u} \in A$ s.t. $\delta J(\bar{u}; v) = 0 \ \forall v \in A^*$ then $\bar{u}$ minimizes J on A [uniquely]

Proof Let $u \in A$. Then $\exists\, v \in A^*$ s.t. $u = \bar{u} + v$. In particular $v = u - \bar{u} \in A^*$.

$$\begin{aligned} J(u) - J(\bar{u}) &= J(\bar{u}+v) - J(\bar{u}) \\ &\geq \delta J(\bar{u}, v) \\ &= 0 \end{aligned}$$

so that $J(u) \geq J(\bar{u})$, $\forall u \in A$. $\square$

Convexity and Integral Functionals

Let $L = L(x, y, z)$ where $y, z \in \mathbb{R}^n$, $x \in \mathbb{R}$.

$$J(y) \equiv \int_a^b L(x, y, y')\,dx$$

For the following $y \in A$ have $y_i(a), y_i(b)$ specified so that $h \in A^*$ vanishes at end points.

Seek conditions on $L(x, y, z)$ so that $J(y)$ is a convex functional:

$$(1) \qquad J(y+h) - J(y) \geq \delta J(y; h)$$

for all $y \in A$, $h \in A^*$. Since

$$\delta J(y;h) = \int_a^b (L_{y_i} h_i + L_{z_i} h_i')\,dx$$

the inequality (1) will be satisfied if the following pointwise constraint is.

$$(2) \quad L(x, y+h, y'+h') - L(x, y, y') - \sum_{i=1}^{n} (L_{y_i} h_i + F_{z_i} h_i') \geq 0$$

Make the identifications

$$p = (y, y') \qquad \Delta p = (h, h')$$

where $p \in \mathbb{R}^{2n}$.

Also define the map $p \mapsto G(x,p)$ by

$$G(x,p) \equiv L(x,y,z)$$

with ∇_p the gradient in $p \in \mathbb{R}^{2n}$.

Supressing the fixed x notation, eqn (2) simplifies to

$$(3) \qquad G(p+\Delta p) - G(p) - \nabla_p G \cdot \Delta p \geq 0$$

By Taylors Theorem $\exists\, \alpha \in [0,1]$ s.t.

$$(4) \qquad G(p+\Delta p) - G(p) - \nabla_p G \cdot \Delta p = \tfrac{1}{2} \Delta p^T H_p(p+\alpha\Delta p)\,\Delta p$$

Thus, if the Hessian matrix H_p is positive definite, (3) will be true and (1) will be satisfied.

<u>Theorem:</u> If $(y,z) \mapsto L(x,y,z)$ is convex $\forall x \in [a,b]$ then any solution of the EL-eqns minimize J on $\mathcal{A}$.

<u>Theorem:</u> If the Hessian H_p of the map $(y,z) \to L(x,y,z)$ is positive definite $\forall x \in [a,b]$ then any solution of the EL-eqns minimize J on $\mathcal{A}$

<u>Remark</u>: For maxima H_p negative definite is a sufficient condition.

EXAMPLE Suppose $L : \mathbb{R}^3 \to \mathbb{R}$, $L = L(x, y, z)$.

$$H_p = \begin{bmatrix} \frac{\partial^2 L}{\partial y^2} & \frac{\partial^2 L}{\partial y \partial z} \\ \frac{\partial^2 L}{\partial z \partial y} & \frac{\partial^2 L}{\partial z^2} \end{bmatrix}$$

In the special case $L_y = 0$ where $L_{y'}$ is a first integral

$$H_p = \begin{bmatrix} 0 & 0 \\ 0 & L_{zz} \end{bmatrix}$$

H_p positive definite $\iff L_{zz} > 0$ for all $x \in [a, b]$

Were

$$L = \sqrt{r(x)^2 + z^2}$$

$$L_{zz} = \frac{r^2}{(r^2 + z^2)^{3/2}} > 0 \qquad \text{(strict)}$$

so (by Theorem) the functional

$$J(y) = \int_a^b \sqrt{r(x)^2 + y'(x)^2}\, dx$$

has a <u>unique</u> minima determined by EL-eqns

Minimal curve length problems, in particular, have unique minima.

Existence and uniqueness Theorems

In what follows we prove some introductory Theorems regarding extrema of

$$I(u) \equiv \int_\Omega L(x, u, Du)\, dx \quad , \qquad Du \equiv \nabla u$$

where $\Omega \subset \mathbb{R}^n$. One needs to know certain definitions regarding Sobolev spaces:

$$W^{k,p}(\Omega) = \{ u \ : \ \|u\|_{W^{k,p}} < \infty \}$$

where the norm

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p}$$

These are subspaces of $L^p(\Omega)$. As an example

$$\|u\|_{W^{1,p}} = \left(\int_\Omega |u|^p + |Du|^p \, dx \right)^{1/p}$$

and if $\Omega = [a, b]$

$$I(u) = \int_a^b L(x, u, u')\, dx$$

EXISTENCE OF A MINIMIZER

Let (1) L be bounded below

(2) L be $C^1(\mathbb{R}^{n+1} \times \Omega)$

(3) $p \mapsto L(p, z, x)$ convex $\forall z \in \mathbb{R}, x \in \Omega$

(4) $L(p, z, x) \geq \alpha |p|^q - \beta$ for some $\alpha > 0, \beta \geq 0, q \in (1, \infty)$

(5) $\mathcal{A} \neq \phi$, $\mathcal{A} = \{ w \in W^{1,q}(\Omega) : w = g \text{ on } \partial\Omega \}$

Then there exists at least one $u \in \mathcal{A}$ such that

$$I(u) = \min_{w \in \mathcal{A}} J(w)$$

<u>Proof</u> (I) First we note the set of real numbers

$$A = \{ z \in \mathbb{R} : z = I(w) \text{ for some } w \in \mathcal{A} \}$$

is bounded below. Hence,

$$m \equiv \inf_{w \in \mathcal{A}} I(w)$$

(since $\mathcal{A} \neq \phi$)
(since L bd below)

exists. Thus, we can select a minimizing sequence $\{u_n\} \subset \mathcal{A}$ with

$$I(u_n) \rightarrow m$$

Moreover, since L is bounded below, i.e.

$$M \leq L(p, z, x)$$

then

$$-\infty < M|\Omega| \leq m$$

yields a finite m.

(II) Without loss of generality $\beta = 0$ else we would consider

$$\tilde{L} \equiv L + \beta$$

Therefore,

$$L(p, z, x) \geq \alpha |p|^q \qquad \text{(coercivity)}$$

$$I(w) \geq \alpha \int_\Omega |Dw|^q dx \qquad \forall w \in \mathcal{A}.$$

Noting then that

$$m \leq I(u_n)$$

$$\alpha \|Du_n\|_{L^q}^q \leq I(u_n)$$

we conclude that

$$\boxed{\sup_n \|Du_n\|_{L^q} < \infty} \quad (*)$$

(III) Fix any $w \in \mathcal{A}$ and note $u_n \in \mathcal{A} \Rightarrow$

$$u_n - w \in W_0^{1,q}(\Omega)$$

Thus

$$\|u_n\|_{L^q} \leq \|u_n - w\|_{L^q} + \|w\|_{L^q} \qquad \text{(PROP OF NORMS)}$$

$$\begin{aligned} &\leq \|(u_n - w) - (\bar{u}_n - \bar{w})\| + \|w\| + \|\bar{u}_n - \bar{w}\| \\ &\leq C_1 \|D(u_n - w)\| + C_n \\ &\leq C_1 \|Du_n - Dw\| + C_2 \\ &\leq C \end{aligned} \Big\} \text{Poincaré.}$$

by (*) above

Conclude

$$\boxed{\sup_n \|u_n\|_{L^q} < \infty}$$

Noting

$$\|u_n\|^q_{W^{1,q}} = \|u_n\|_{L^q} + \|Du_n\|_{L^q}$$

we find

$$\boxed{\{u_n\} \text{ bounded in } W^{1,q}(\Omega)}$$

<u>PART IV</u> Since $W^{1,q}(\Omega)$ a Hilbert space and $\{u_n\}$ bounded, $\exists \{u_{n_k}\} \ni$

$$u_{n_k} \rightharpoonup u \in W^{1,q} \quad \text{weakly.}$$

In view of the fact that assumptions (1)–(3) imply I is weakly lower semi continuous

$$I(u) \le \liminf_{k\to\infty} I(u_{n_k}) = m$$

But $u \in \mathcal{A}$ implies

$$I(u) = \min_{w \in \mathcal{A}} I(w) \qquad \square.$$

UNIQUENESS OF MINIMIZERS

Should not expect all $I(w)$ to have unique minima.

Let $L \geq 0$ and z_k be roots of L in z argument.

$$L(p, z_k, x) = 0 \qquad \forall p \in \mathbb{R}^n,\ x \in \Omega$$

Then $I(z_k) = 0 = \min\limits_{w \in \mathcal{A}} I(w)$ over

$$\mathcal{A} = \{ w \in W^{1,q} : w = 0 \text{ on } \partial\Omega \}$$

as one of many examples.

THEOREM (UNIQUENESS)

(1) $L \in C^2(\mathbb{R}^{n+1} \times \Omega)$
(2) $L = L(p, x)$
(3) $p \mapsto L(p, x)$ is strictly convex

Then minimizers are unique.

PROOF Let $u, \tilde{u}$ be any two minimizers
Since L is convex stictly and L is C^2

$$L_{p_i p_j}(p, x)\, z_i z_j \geq \theta |z|^2 > 0 \qquad z \neq 0,\ z \in \mathbb{R}^n$$

and $\theta > 0$. Thus, $\forall p, q \in \mathbb{R}^n$, $x \in \Omega$

$$(1) \quad L(p, x) \geq L(q, x) + D_p L(q, x) \cdot (p - q) + \frac{\theta}{2} |p - q|^2$$

by Taylors theorem

Let $v = \frac{1}{2}(u+\tilde{u})$.

Set $\underline{p = Du}$, $q = Dv$ in (1) and integrate

$$(2) \quad I(u) \geq I(v) + \int D_pL(v,x)\cdot\left(\frac{Du - D\tilde{u}}{2}\right)dx + \frac{\theta}{8}\int |Du - D\tilde{u}|^2\,dx$$

Set $\underline{p = D\tilde{u}}$, $q = Dv$. in (1) and integrate

$$(3) \quad I(\hat{u}) \geq I(v) + \int D_pL(v,x)\cdot\left(\frac{D\tilde{u} - Du}{2}\right)dx + \frac{\theta}{8}\int |Du - D\tilde{u}|^2\,dx$$

Add (2)-(3) and divide by 2 to conclude

$$\boxed{\frac{I(u) + I(\tilde{u})}{2} \geq I(v)}$$

More accurately,

$$(4) \qquad I(v) + \frac{\theta}{8}\int |Du - D\tilde{u}|^2\,dx \leq \frac{1}{2}\left(I(u) + I(\hat{u})\right) = m$$

But, since

$$m = I(u) = I(\tilde{u}) = \min_{w \in \mathcal{A}} I(w) \leq I(v)$$

we deduce from (4) that $Du = D\tilde{u}$ a.e. Since $u, \tilde{u} = g$ on $\partial\Omega$ conclude

$$u = \tilde{u} \qquad \text{a.e.} \qquad \square$$