Geodesics in $\mathbb{R}^3$

Let surface S have parametrization

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

Of all paths Γ on S connecting P_1 and P_2 on S, which has the shortest length? Such paths are called geodesics.

Parametrize path $r(t) = (X(t), Y(t), Z(t))$

$$X(t) = x(u(t), v(t))$$
$$Y(t) = y(u(t), v(t))$$
$$Z(t) = z(u(t), v(t))$$

Arclength

(1) $$L = \int_{t_1}^{t_2} (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)^{1/2}\, dt$$

where

(2) $$\dot{X} = x_u \dot{u} + x_v \dot{v}$$

(3) $$\dot{Y} = y_u \dot{u} + y_v \dot{v}$$

(4) $$\dot{Z} = z_u \dot{u} + z_v \dot{v}$$

Since $\dot{X}, \dot{Y}, \dot{Z}$ are functions of $(u, v, \dot{u}, \dot{v})$ the integrand is as well.

Recall $\|\dot{r}\| = \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}$. Using (2)-(3)

$$(4) \qquad \|\dot{r}\| = \sqrt{P\dot{u}^2 + 2Q\dot{u}\dot{v} + R\dot{v}^2} = L(u, v, \dot{u}, \dot{v})$$

where

$$P = x_u^2 + y_u^2 + z_u^2$$

$$Q = x_u x_v + y_u y_v + z_u z_v$$

$$R = x_v^2 + y_v^2 + z_v^2.$$

In summary

$$J(u,v) = \int_{t_1}^{t_2} L(u, v, \dot{u}, \dot{v})\, dt$$

must be minimized. Here the Lagrangian depends on two functions and the EL-eqns are

$$L_u = \frac{d}{dt} L_{\dot{u}}$$

$$L_v = \frac{d}{dt} L_{\dot{v}}.$$

The B.C. are $(u(t_k), v(t_k)) = (U_k, V_k)$ at $k=1,2$. Once EL eqns are solved with these B.C. the path coordinates can be reconstructed from $X(t) = x(u(t), v(t)), \ldots$

Metric Tensor

Let surface $\mathcal{S}$ have parametrization

$$x = x(s) \qquad y = y(s) \qquad z = z(s)$$

where $s = (s_1, s_2)$. Next let Γ be a curve on $\mathcal{S}$ given by

$$X = x(s(t)) \qquad Y = y(s(t)) \qquad Z = z(s(t))$$

and let $r(t) = (X(t), Y(t), Z(t))$. Then for some P, Q, R

$$(1) \qquad \| \dot{r}(t) \| = \sqrt{P \dot{s}_1^2 + 2Q \dot{s}_1 \dot{s}_2 + R \dot{s}_2^2}$$

From previous development P, Q, R are functions of (s_1, s_2). Thus $\exists g(s)$ s.t.

$$L = \int_{t_1}^{t_2} \| \dot{r}(t) \| dt = \int_{t_1}^{t_2} \sqrt{g_{ij} \dot{s}_i \dot{s}_j} \, dt$$

where g_{ij} is the <u>metric tensor</u> where sum notation (repeated index $\Rightarrow$ sum) implies

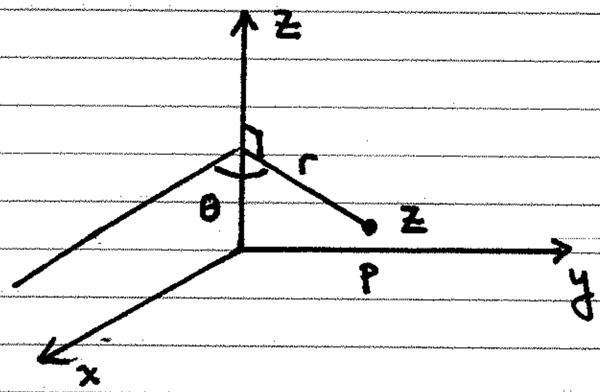
$$g_{ij} \dot{s}_i \dot{s}_j = g_{11} \dot{s}_1 \dot{s}_1 + g_{12} \dot{s}_1 \dot{s}_2 + g_{21} \dot{s}_2 \dot{s}_1 + g_{22} \dot{s}_2 \dot{s}_2$$

or

$$g_{ij} \dot{s}_i \dot{s}_j = (\dot{s}_1, \dot{s}_2) \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix}$$

EXAMPLE Cylindrical Coordinates

Suppose a surface can be described by a graph $z = f(x,y)$.



In polar coordinates $(u,v) = (r,\theta)$ and the surface parametrization is

$$x = x(r,\theta) = r\cos\theta$$

$$y = y(r,\theta) = r\sin\theta$$

$$z = z(r,\theta) = f(r\cos\theta, r\sin\theta)$$

An explicit example would be

$$z = f(x,y) = x^2 + y^2 + x$$

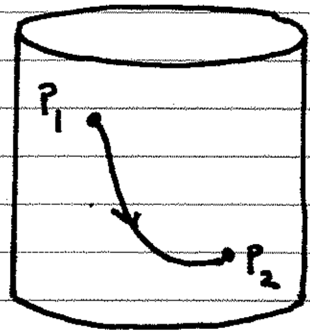
in which case

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = r^2 + r\cos\theta$$

EXAMPLE Geodesics on a cylinder



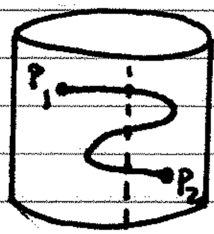
Use polar coordinates

$$x = a\cos\theta$$

$$y = a\sin\theta$$

$$z = z(\theta)$$

where a is the radius. We have tacitly assumed that z is a function of θ which need not be the case as the following figure illustrates.



3 different z values for same θ

Under this simplifying reduction

$$r(\theta) = (a\cos\theta, a\sin\theta, z(\theta))$$

$$r'(\theta) = (-a\sin\theta, a\cos\theta, z'(\theta))$$

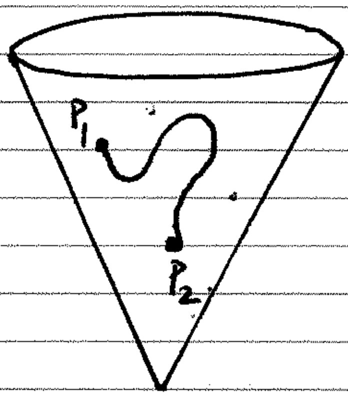
Computing $\|r'(\theta)\|$ we find the arclength functional

$$J(z) = \int_{\theta_1}^{\theta_2} \sqrt{a^2 + z'(\theta)^2}\, d\theta$$

$$\mathcal{A} = \{ z \in C^2[\theta_1, \theta_2] : z(\theta_k) = z_k \}$$

The solution has $z(\theta)$ linear in θ, as expected.

EXAMPLE Geodesics on a cone $x^2 + y^2 = a^2 z^2$



The surface in polar is parametrized by

$$x = x(\theta, r) = r\cos\theta$$

$$y = y(\theta, r) = r\sin\theta$$

$$z = z(\theta, r) = ar$$

For the path above neith r nor z are functions of θ. We shall first formulate the problem for this case and then assume $r = r(\theta)$, an intuitive assumption.

Let $r = r(t)$, $\theta = \theta(t)$. Then

$$\dot{X} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$\dot{Y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$
$$\dot{Z} = a\dot{r}$$

The Lagrangian $L = \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}$ then simplifies

$$J(r, \theta) = \int_{t_1}^{t_2} \sqrt{r^2\dot{\theta}^2 + (1+a^2)\dot{r}^2}\, dt$$

The resulting EL-eqns are difficult to solve but they have one first integral since

$$L_\theta = 0$$

If we now assume $r = r(\theta)$ on extrema

$$x(\theta) = r(\theta)\cos(\theta)$$

$$y(\theta) = r(\theta)\sin(\theta)$$

$$z(\theta) = a\,r(\theta)$$

The arclength Lagrangian is, now,

$$L = \sqrt{(x')^2 + (y')^2 + (z')^2} \qquad (\ )' = \frac{d}{d\theta}(\ )$$

Simplifying this

$$J(r) = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (1+a^2)(r')^2}\, d\theta$$

$$\mathcal{A} = \{ r \in C^2[\theta_1, \theta_2] : r(\theta_k) = r_k \}$$

Solving the Euler Lagrange eqns.

Since $L_\theta = 0$ we have the first integral

$$(1) \quad L - r' L_{r'} = \sqrt{r^2 + b^2 r'^2} - \frac{r'^2 b^2}{\sqrt{r^2 + b^2 r'^2}} = k$$

for some constant k.

One method is to solve

$$L - r' L_{r'} = k$$

for r' to get

$$\frac{dr}{d\theta} = \frac{r\sqrt{r^2 - k^2}}{kb}$$

which is separable. The resulting integrals are doable but very messy.

A rather brilliant solution in some books involves the observation that the term $\sqrt{r^2 + b^2 r'^2}$ looks like an inverse trig sub. Letting

$$(2) \qquad r = c_1 \sec\left(\frac{\theta}{b} + c_2\right)$$

and using $\tan^2 x + 1 = \sec^2 x$ and (2) in (1) we find (1) is satisfied $\forall \theta$ if $k = c_1$

Therefore (2) is the solution. c_k must be chosen to satisfy B.C.