# Hamiltonian Mechanics

The principle of least action states that the path followed by a system is that which extremizes the action

$$J(q) = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

where the Lagrangian $L = T - U$. This results in the equations of motion:

$$(1) \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

These are $n$ coupled $2^{nd}$ order differential equations for $q_i(t)$.

Recall the momenta $p_i$ conjugate to $q_i$ are defined by

$$(2) \qquad p_i \equiv \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}, t) \qquad i = 1, \ldots n$$

This should be regarded as a set of $n$ coupled (algebraic) equations defining a transformation

$$p \leftrightarrow \dot{q}$$

In Hamiltonian mechanics one views the soln (motion) as one in $(q, p)$ phase space and reformulates (1).

Given the definition of $p_i$ the EL-eqns (1):

$$(3) \qquad \frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}(q, \dot{q}, t)$$

In so much as (2) yields an invertible transformation $p \leftrightarrow \dot{q}$ then the right side of (3) is a function of $(q, p)$.

## Hamiltonian Definition

$$(4) \qquad H \equiv \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \dot{q}_i p_i - L$$

## Conservation of H. (Assume $L_t = 0$)

$$(5) \qquad \frac{d}{dt}\left(\dot{q}_i p_i - L\right) = \cancel{\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}} + \cancel{\dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}} - \cancel{\frac{\partial L}{\partial q_i}\dot{q}_i} - \cancel{\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i}$$

hence

$$\frac{dH}{dt} = 0$$

is a constant of the motion.

If $L_t \neq 0$ then similar calculates yield

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

## Hamiltonian and Energy

If $x$ are the cartesian coordinates for a system and $q$ the generalized coordinates, there exists $T_{ij}(q)$ such that the kinetic energy

$$T(q,\dot{q}) = \sum_{i,j} T_{ij}(q)\,\dot{q}_i \dot{q}_j$$

**EX** For the spherical pendulum

$$T = \tfrac{1}{2} m R^2 \left( \sin^2\phi\, \dot{\theta}^2 + \dot{\phi}^2 \right)$$

wlog $mR^2 = 2$ so that for $q = (\theta, \phi)$

$$[T_{ij}] = \begin{bmatrix} \sin^2\phi & 0 \\ 0 & 1 \end{bmatrix}$$

**Defn** A function $F(q,\dot{q})$ is homogeneous of degree $n$ if

$$F(q, \alpha\dot{q}) = \alpha^n F(q,\dot{q}) \qquad \forall \alpha \in \mathbb{R}$$

**Lemma** Since $T(q,\dot{q})$ is homogeneous of degree $n=2$

$$\dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = nT$$

**Pf**: Define $F(\alpha) = T(q, \alpha\dot{q})$ and compute $F'(1)$

$$F'(1) = \underbrace{\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i}_{\text{chain rule}} = \underbrace{n\,T(q,\dot{q})}_{\text{defn of homog}} \qquad \square$$

<u>Theorem</u> If $\frac{\partial U}{\partial \dot{q}_i} = 0$ then $H = T + U$

Pf/

$$H \equiv \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$\frac{\partial U}{\partial \dot{q}_i} = 0$$

$$H = \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - L$$

Lemma

$$H = 2T - L$$

$$H = T + U \qquad \square$$

<u>Summary</u>

(1) $$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

(2) $$H = T + U \quad \text{if } U = U(q)$$

Remarking if $L_t = 0$ then $H$ conserved.

## Hamiltons Equations

Seek eqns for $(q,p) \in \mathbb{R}^{2n}$ that extremize the action

$$(1) \qquad J(q) = \int_{t_1}^{t_2} \underbrace{(p_i \dot{q}_i - H(q,p))}_{L} \, dt$$

where the system

$$(2) \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q})$$

(2) defines a transformation $(q,\dot{q}) \leftrightarrow (q,p)$. As such $\dot{q}_i$ may be viewed as function of $(q,p)$. Now note

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

Using the EL-eqns and (2)

$$dL = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) dq_i + p_i \, d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

or

$$dL = \dot{p}_i \, dq_i + \underline{p_i \, d\dot{q}_i} + \frac{\partial L}{\partial t} dt$$

$$(3) \qquad \underset{\uparrow}{dL} = \dot{p}_i \, dq_i + \underset{\uparrow}{\underbrace{d(p_i \dot{q}_i) - \dot{q}_i \, dp_i}} + \frac{\partial L}{\partial t} dt$$

Indicated terms in (3) can be written as a differential of the Hamiltonian

Rearranging

$$d\big(\underbrace{p_i\dot{q}_i - L}_{H}\big) = -\dot{p}_i\, dq_i + \dot{q}_i\, dp_i - \frac{\partial L}{\partial t}\, dt$$

Hence

$$\boxed{dH = \frac{\partial H}{\partial q_i}\, dq_i + \frac{\partial H}{\partial p_i}\, dp_i = -\dot{p}_i\, dq_i + \dot{q}_i\, dp_i - \frac{\partial L}{\partial t}\, dt}$$

Matching differentials we arrive at Hamilton's equations for the motion

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \frac{dH}{dt} &= -\frac{\partial L}{\partial t} \end{aligned}}$$

Again note $H$ is conserved if $L_t = 0$.

## Alternate derivations

We shall accept the first of Hamilton's equations (hard to prove)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

and derive the second equation for $\dot{p}_i$

For clarity we define $(q, \dot{q}) \leftrightarrow (Q, P)$ by

$$Q_i = q_i \qquad P_i = \frac{\partial L}{\partial \dot{q}_i}(Q, \dot{q})$$

so that $H = H(Q, P)$. Then

$$\frac{\partial H}{\partial q_i} = \frac{\partial H}{\partial P_K}\frac{\partial P_K}{\partial q_i} + \frac{\partial H}{\partial Q_i}$$

$$= -\dot{q}_K \frac{\partial P_K}{\partial q_i} + \frac{\partial H}{\partial Q_i}$$

Defn of $H = P_i \dot{q}_i - L$ and chain rule.

$$= \frac{\partial}{\partial q_i}\left(H - \dot{q}_K P_K\right)$$

$$= -\frac{\partial L}{\partial q_i}$$

$$= -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

EL-eqns

$$= -\dot{P}_i$$

Hence

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i}$$

EXAMPLE Cartesian coordinates $x = (x_1, x_2, x_3)$

Derive Hamiltons equations for a particle of mass $m$ with potential $\mathcal{U} = \mathcal{U}(x)$.

$$T = \tfrac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) = \tfrac{1}{2} m \|\dot{x}\|^2$$

Determine conjugate momenta from Lagrangian

$$L = T - U = \tfrac{1}{2} m (\dot{x}_i)^2 - U(x)$$

Since $U_{\dot{x}_i} = 0$ the Hamiltonian is $H = T + U$

$$p_i \equiv \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i$$

So that

$$\dot{x}_i = \frac{1}{m} p_i$$

and

$$H = \frac{1}{2m} (p_1^2 + p_2^2 + p_3^2) + U(x)$$

Hamilton's eqns are

$$(1) \quad \dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{1}{m} p_i$$

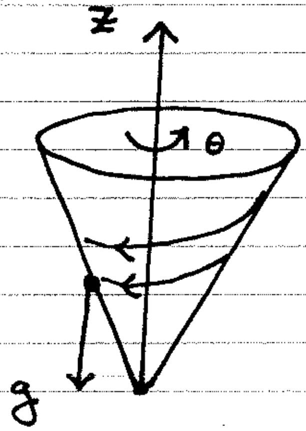
$$(2) \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial U}{\partial x_i}$$

The second equation implies $\dot{p} = -\nabla U$. The first yields the standard defn of momentum $p_i = m\dot{x}_i$. Collectively

$$\frac{d}{dt}(m\dot{x}_i) = -\frac{\partial U}{\partial x_i}$$

$$\frac{dp}{dt} = -\nabla U$$

**EXAMPLE** A bead of mass m slides frictionlessly on a cone $az = \sqrt{x^2+y^2}$ where $a \in \mathbb{R}$




Choose cylindrical coordinates as the generalized coordinates.

$$q = (r, \theta, z)$$

The problem has a holonomic constraint

$$(1) \qquad r = az$$

We seek to find Hamilton's eqns for motion. Need to express $T = T(q, \dot{q})$ and then find p from EL-eqns. Use (1) and

$$x = r\cos\theta = az\cos\theta$$
$$y = r\sin\theta = az\sin\theta$$

Then after some calculations

$$T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \tfrac{1}{2}m\left((1+a^2)\dot{z}^2 + a^2 z^2\dot{\theta}^2\right)$$

Lagrangian is

$$(1) \qquad L = \tfrac{1}{2}\left[(1+a^2)\dot{z}^2 + a^2 z^2 \dot{\theta}^2\right] - mgz$$

where gravitational $\mathcal{U} = mgz$.

## Conjugate momenta $p = (p_z, p_\theta)$

$$(2) \qquad p_z \equiv \frac{\partial L}{\partial \dot z} = m(1+a^2)\dot z$$

$$(3) \qquad p_\theta \equiv \frac{\partial L}{\partial \dot\theta} = m a^2 z^2 \dot\theta$$

Here $q = (z, \theta)$ and $\dot q = (\dot z, \dot\theta)$. Eqns (2)-(3) define a transformation $(\dot q, q) \leftrightarrow (p, q)$.

Since $\mathcal{U}_{\dot q_i} = 0$, $H = T + \mathcal{U}$. Since $T$ is computed already, we use (2)-(3) to eliminate $(\dot z, \dot\theta)$ to get

$$(4) \quad H = T + \mathcal{U} = \frac{p_z^2}{2m(1+a^2)} + \frac{p_\theta^2}{2ma^2z^2} + mgz$$

Hence we get the four Hamiltons eqns

$$(5) \qquad \dot p_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$(6) \qquad \dot p_z = -\frac{\partial H}{\partial z} = -mg + \frac{p_\theta^2}{ma^2z^3}$$

$$(7) \qquad \dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ma^2z^2}$$

$$(8) \qquad \dot z = \frac{\partial H}{\partial p_z} = \frac{p_z}{m(1+a^2)}$$

$\ddot z$ eqn

## Conserved Quantities

$\dot p_\theta = 0$ — eqn (5) above

$\dot H = 0$ — $\frac{\partial L}{\partial t} = 0$

Values of these constants found from $q(0), \dot q(0)$ in (2)-(3) and defn of $H$.

## Solving the eqns of motion.

Differentiate eqn (8) in $t$ and use (6):

$$(9) \qquad \ddot{z} + F(z) = 0$$

where $F(z) = \alpha - \beta z^{-3}$ for $\alpha, \beta > 0$ given by

$$\alpha = \frac{g}{1+a^2} \qquad \beta = \frac{p_\theta^2}{m^2 a^2 (1+a^2)}$$

Note $\beta$ is constant.

Eqns like (9) frequently arise in mechanics.

$$\dot{z}\ddot{z} + F(z)\dot{z} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{z}^2 + f(z)\right) = 0 \qquad f(z) = \alpha z + \frac{1}{2}\beta z^{-2}$$

$$(10) \qquad \frac{1}{2}\dot{z}^2 + f(z) = E$$

for some constant $E$. This eqn is not a surprise since $H$ is constant. Can re-derive (10) using (8) in (4) to eliminate $p_z$.

Basically, there are four eqns of motion and two constants of the motion so one expects a reduction to a single 2nd equation like (10). Solving (10) amounts to the following quadrature

$$\int \frac{z\,dz}{\sqrt{2Ez^2 - 2\alpha z^3 + \beta}} = dt \qquad \text{(sadly not doable)}$$

## EXAMPLE Central Forces

In spherical coordinates $q = (r, \theta, \phi)$ where $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$.

$$L = T - U = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \sin^2\theta \, \dot{\phi}^2 - U(r)$$

Conjugate momenta

$$(1) \qquad p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}$$

$$(2) \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$(3) \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi} \qquad \text{(conserved)}$$

Note that (1)-(3) define $(q, \dot{q}) \leftrightarrow (q, p)$

Since $L_\phi = 0$ the EL-eqns $\Rightarrow p_\phi$ is constant. And, since $L_t = 0$, the previous theorem implies the Hamiltonian is also Constant. Moreover $\frac{\partial U}{\partial \dot{q}_i} = 0$ implies $H = T + U$

Using (1)-(3) to eliminate $\dot{q}$ we find

$$H = \frac{1}{2m} \left\{ p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right\} + U(r)$$

Just to be explicit we list all six of Hamilton's eqns

$$(1) \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$(2) \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$(3) \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2\theta}$$

and associated momenta eqns

$$(4) \quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2\theta} - \frac{\partial \mathcal{U}}{\partial r}$$

$$(5) \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta}$$

$$(6) \quad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

Phase space $(q, p) \in \mathbb{R}^6$.

Note that for obital motion on $\phi = \phi_0$ we have $\dot{\phi} = 0$, $p_\phi = 0$ and indicated terms vanish. Then $p_\theta$ is constant and (1), (4) yield (for constant m)

$$m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{\partial \mathcal{U}}{\partial r} = F(r)$$

which in principle is integrable.

EXAMPLE Velocity dependent potential

The Lorenz force acting on a particle of charge e under the influence of electric field $\vec{E}$ and magnetic field $\vec{B}$ is

$$(1) \qquad \vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

where $\vec{v}$ is velocity (cartesian).

From the theory of electromagnetism

$$(2) \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

where

$$\phi = \text{scalar electric potential}$$

$$\vec{A} = \text{vector magnetic potential}$$

The associated "potential energy" is for cartesian coordinates q

$$V(q, \dot{q}, t) = e(\phi - \dot{q}_i A_i) \qquad *$$

with the resulting Lagrangian

$$\boxed{\begin{aligned} L &= \tfrac{1}{2} m \dot{q}_i^2 - e(\phi - \dot{q}_i A_i) \\ L &= \tfrac{1}{2} m \|\dot{q}\|^2 - e\phi + e\dot{q} \cdot A \end{aligned}}$$

* show this yields (1) via EL eqns for $A_t = 0$, $A_{\dot{q}} = 0$

As long as $L_t = 0$ the Hamiltonian is still conserved. However $V$ depends on $\dot{q}$ so the theorem which assures $H = T + V$ is no longer valid and one must use

$$(3) \qquad H(q,p) \equiv \dot{q}_i p_i - L(q,\dot{q},t)$$

where conjugate momenta are

$$(4) \qquad p_i = \frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i + eA_i$$

or in vector form

$$p = m\dot{q} + eA$$

We use (4) to eliminate $\dot{q}_i$ in (3). After some calculations one obtains a simple vector form the Hamiltonian. (* show)

$$\boxed{H = \frac{1}{2m}(p - eA)\cdot(p - eA) + e\phi}$$

This is the correct form for quantum mechanics.

Hamiltons equations are a tad messy.

# Canonical Transformations

Is any transformation $(p, q, t) \leftrightarrow (P, Q, t)$ for which Hamiltons equations remain invariant

$$h(p, q, t) = \text{old Hamiltonian}$$

$$H(P, Q, t) = \text{new Hamiltonian}$$

Define old and new actions

$$s(p, q) = \int (p_i \dot{q}_i - h)\, dt$$

$$S(P, Q) = \int (P_i \dot{Q}_i - H)\, dt$$

Note $\delta s = 0$ and $\delta S = 0$ simultaneously if the actions differ by an exact differential, or that $\exists F$ s.t.

$$(1) \qquad p_i \dot{q}_i - h = P_i \dot{Q}_i - H + \frac{dF}{dt}$$

Many choices but one is

$$(2) \qquad F = F_2(q, P, t) - P_i Q_i$$

For such $F$, eqn (1) is true if

$$\left.\begin{array}{ll} (3) & p_i = \dfrac{\partial F_2}{\partial q_i}(q, P, t) \\ (4) & Q_i = \dfrac{\partial F_2}{\partial P_i}(q, P, t) \end{array}\right\} \text{implicitly define } (q, p) \leftrightarrow (Q, P)$$

$$(5) \qquad H = h + \frac{\partial F_2}{\partial t}$$

# Hamilton Jacobi Theory

Seek a canonical transformation so the new Hamiltonian $H \equiv 0$.
In that case

$$\dot{Q}_i = 0 \qquad \dot{P}_i = 0$$

so $Q_i, P_i$ are $2n$ constants of the motion.
In doing so, one would have solved the equations of motion.

Using (5) and (3) the transformation which accomplishes this goal satisfies the Hamilton-Jacobi eqn (PDE)

$$\boxed{H = h\left(q_1, q_2, \ldots, \frac{\partial F_2}{\partial q_1}, \ldots \frac{\partial F_2}{\partial q_n}, t\right) + \frac{\partial F_2}{\partial t} = 0}$$

where $p_i = \alpha_i$ constant. Any soln will do.

EXAMPLE $\quad h(q,p) = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$ Harm. Osc.

Hamilton Jacobi equation $F_2 = S$ is

$$\frac{1}{2m}\left(\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right) + \frac{\partial S}{\partial t} = 0$$

Letting $S(q,\alpha,t) = W(q,\alpha) - \alpha_1 t$ yields

$$\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2 = 2m\alpha_1$$

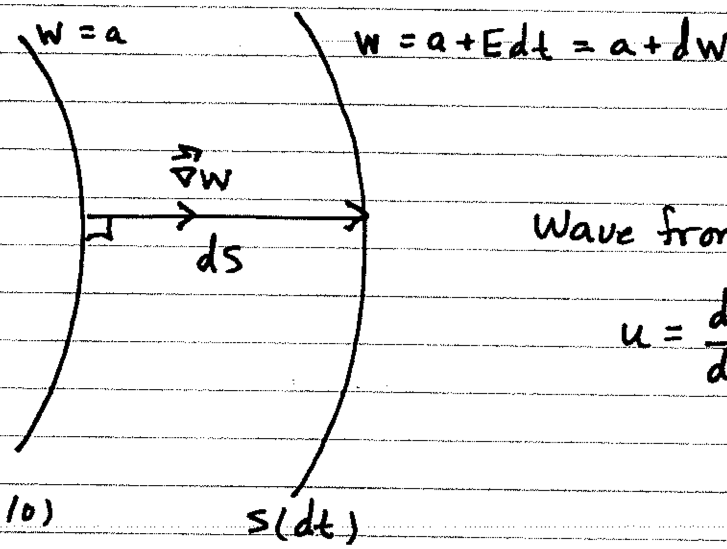
which is a separable, solvable first order nonlined ODE for W.

## Generating functions and waves

Suppose $H$ is conserved and $H = E$ is the energy. Let $S(q, P, t)$ be the generating function that solves the Hamilton Jacobi Eqn. (HJ)

$$S(q, P, t) \equiv W(q, P) - Et$$

Then $W(q, P) = \text{constant}$ are surfaces in $(p, q)$ phase space.




Wave front velocity

$$u = \frac{ds}{dt}$$

Hamilton Jacobi equation for one-particle

$$\frac{1}{2m}|\nabla W|^2 + V = E$$

where $E$ = energy and $V$ = potential energy.

Since $p_i = \dfrac{\partial W}{\partial q_i}$.

## Schrödinger Eqn

$$\frac{\hbar^2}{2m}\nabla^2\psi - V\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial t}$$

If one lets

$$\psi(q,t) = \psi_0 \exp(iS/\hbar)$$

you get

$$\underbrace{\frac{1}{2m}(\nabla S)^2 + V + \frac{\partial S}{\partial t}}_{\text{Very close. Equal to HJ if (RHS)}=0\text{ !!}} = \frac{ik\,\nabla^2 S}{2m}$$

when does this happen?

$$\hbar\,\nabla^2 S \ll |\nabla S|^2$$

But $p_i = \dfrac{\partial S}{\partial q_i}$ so same as

$$\hbar\,\nabla\cdot p \ll p^2$$

$$\frac{\hbar}{p}\frac{dp}{dx} \ll 1 \qquad \text{one dim.}$$