Calculus of Variations

Let V be some vector space of real valued functions. Now let J be a functional

$$J : A \to \mathbb{R}$$

where $A \subset V$ is a subset (not necessarily subspace) of admissible functions $u \in A$. Goal is to minimize J over A

$$\min_{u \in A} J(u)$$

In some texts if the min is attained by some $\bar{u} \in A$ one writes

$$\bar{u} = \underset{u \in A}{\operatorname{argmin}} J(u) .$$

Key theoretical issues

(1) Necessary conditions on solns

(2) Existence and uniqueness of solns

(3) Sufficient conditions

In practice the sets A may be very complex and defined by several constraints

EX Point evaluation functional

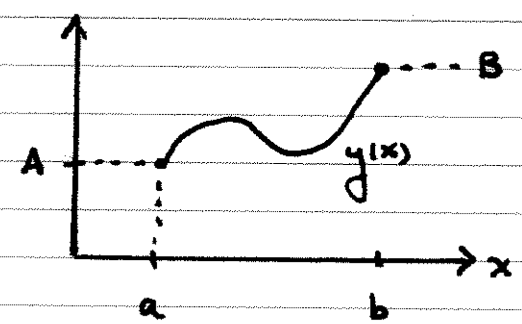
$$J(y) = y(1)^2 \qquad \mathcal{A} = C[0,2]$$

EX Area functional

$$J(y) = \int_0^1 y(x)\,dx$$

A possible constraint might be $y(x) \geq 0$

EX Arclength functional



$V = PC^1[a,b]$

piecewise C^1

$$J(y) = \int_a^b \sqrt{1 + y'^2}\,dx$$

Here the admissible set

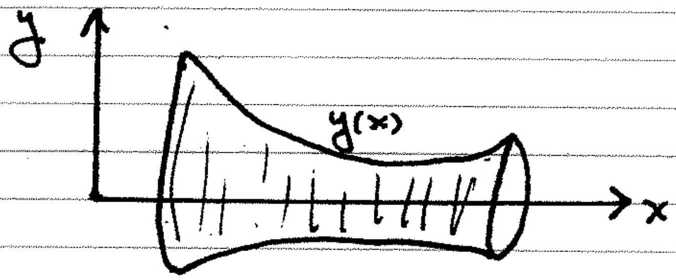
$$\mathcal{A} = \{ y \in V \;:\; y(a) = A,\ y(b) = B \}$$

and is clearly not a subspace of V.

Also the minimizer is the straight line connecting endpoints

$$\bar{y}(x) = B + \frac{(B-A)}{(b-a)}(x-a)$$

EX Surface area of revolution



$y(a) = A$

$y(b) = B$

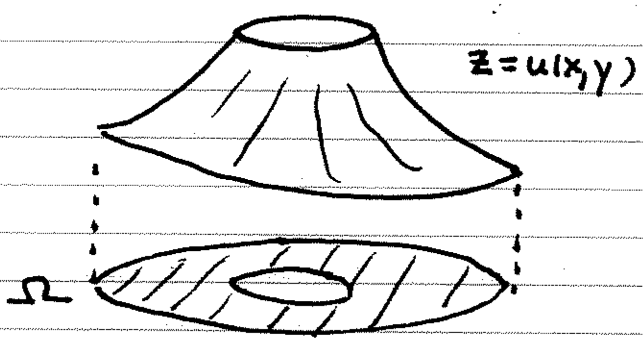
From elementary calculus the surface area of revolution is

$$J(y) = \int_a^b 2\pi y \sqrt{1 + y'^2}\, dx$$

Surprisingly the answer is NOT part of a cone.

EX Minimum Area

$V = C^1(\Omega)$



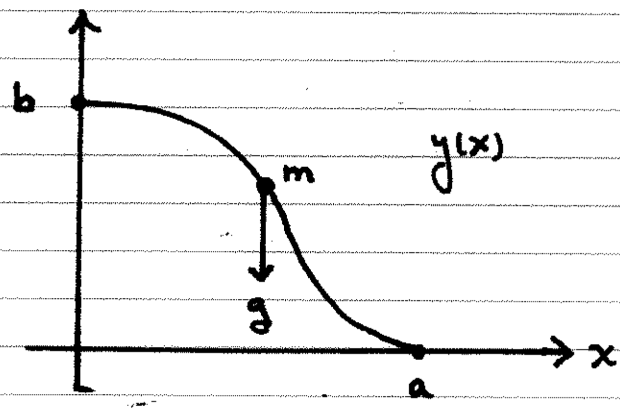
Given Ω and the value of u on the boundary $\partial\Omega$ we seek the surface of minimum area

$$J(u) = \int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dA$$

where

$$A = \{ u \in C_1(\Omega) \ : \ u\big|_{\partial\Omega} = g \}$$

Ex Brachistochrone Problem



A bead of mass m slides down a frictionless wire at speed v under the influence of gravity. Of all shapes $y(x)$, which minimizes the transit time T.

$$(1) \quad T = \int_0^T dt = \int_0^L \frac{dt}{ds}\, ds = \int_0^L \frac{1}{v}\, ds = \int_0^a \frac{\sqrt{1+y'^2}}{v}\, dx$$

Use conservation of energy to find v

$$(2) \quad mgb = \frac{1}{2}mv^2 + mgy$$

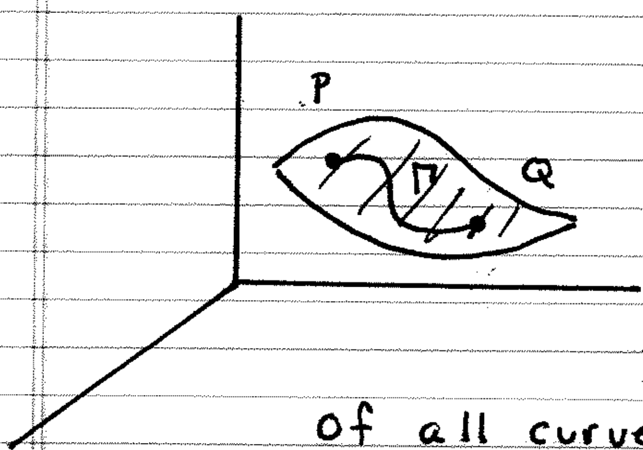
Solve (2) for v to use in (1). Must minimize

$$T(y) \equiv \int_0^a \sqrt{\frac{1+y'^2}{2g(b-y)}}\, dx$$

over admissible set

$$A = \{ y \in C^2[0,a] : y(0) = b,\ y(a) = 0 \}$$

EX Geodesics



$g(x,y,z)=0$
surface

Of all curves Γ on $g=0$, which has the shortest length. Such curves are called geodesics

$$\mathbf{X}(t) = (x(t), y(t), z(t)) \qquad t \in [0,T]$$

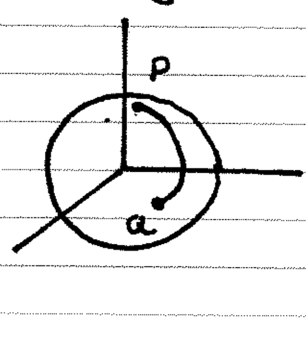
Seek to minimize

$$J(\mathbf{X}) = \int_0^T \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

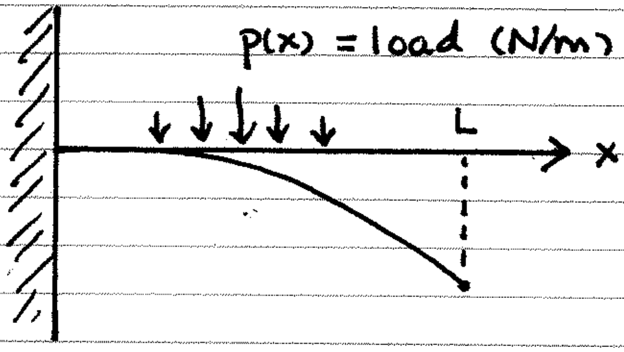
over an admissible set

$$A = \{ \mathbf{X} : \mathbf{X}(0) = P, \mathbf{X}(T) = Q, g(\mathbf{X}) = 0 \}$$

Note geodesics on spheres are "great circles"



Ex Beam deflection problem



Minimize potential energy

$$V(y) = \int_0^L [\tfrac{1}{2}\mu y''^2 - p(x) y]\,dx$$

where μ = flexural rigidity (elastic component of pot. energy)

$$\min_{y \in A} V(y)$$

where

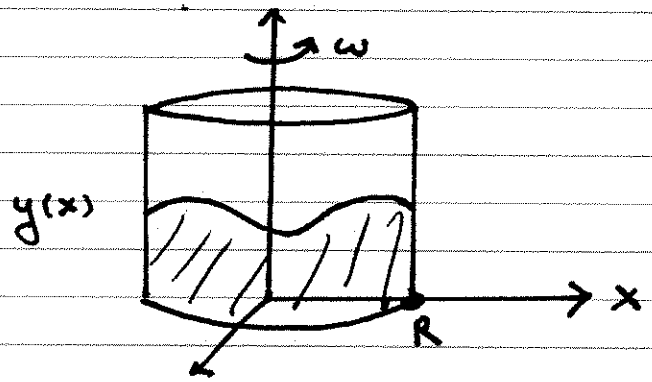
$$A = \{ y \in C^4[0,L] : y(0) = y'(0) = 0 \}.$$

Note boundary conditions at $x = L$ are "free".

Part of minimization procedure is to find the B.C. at $x = L$.

Ex Rotating Fluid (Min Pot. Energy)

A fluid of density ρ in a cylinder of radius R is rotated with angular velocity ω



Potential Energy

- static
- due to centrifugal energy.

$$J(y) = \rho\pi \int_0^R [\underbrace{gy^2}_{\text{gravity}} - \underbrace{\omega x^2 y}_{\text{centrifugal}}]\, x\, dx$$

The integral arises from using the method of cylindrical shells to sum up contribution for P. Energy.

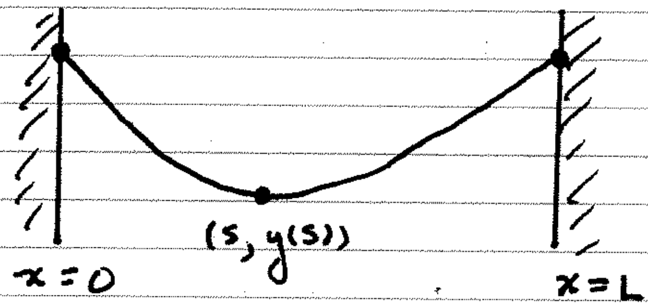
Has an "<u>isoperimetric</u>" constraint defining $\mathcal{A}$

$$V(y) = 2\pi \int_0^R xy\, dx \;=\; \text{volume } V_0$$

Since volume of fluid is fixed

$$\mathcal{A} = \{ y \in C[0,R] : V(y) = V_0 \}$$

EX Hanging cable problem



Let s be arclength (s = 0 at x = 0) and L_c be the length of the cable with $L_c > L$.

Shape is that which minimizes the potential energy

$$J(y) = \mu \int_0^{L_c} y(s)\,ds \qquad \text{pot. energy}$$

where μ = weight/length. Span fixed

$$K(y) = \int_0^{L} dx = \int_0^{L_c} \sqrt{1 + y'(s)^2}\,ds$$

follows from $x'(s)^2 + y'(s)^2 = 1$ (unit tangent)

Seek y(s) to solve

$$\min_{y \in A} J(y)$$

where

$$A = \{ y : y(0) = y(L_c) = 0,\ K(y) = L \}$$

Ex Classical Mechanics (nonrelativistic motion)

Let $q \in \mathbb{R}^n$ be coordinates of a system of particles and $\dot{q} = \frac{dq}{dt}$.

$$J(q) \equiv \int_0^t (T(q,\dot{q}) - U(q,\dot{q}))\,dt$$

is the "action" of the system when

$T(q,\dot{q})$ = total kinetic energy

$U(q,\dot{q})$ = total potential energy

In physics the principle of least action states the physically realized path is that which extremizes (minimizes) J.

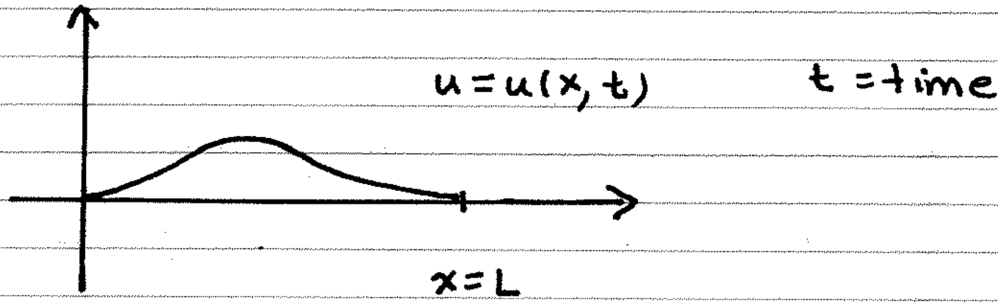
Note the action J is not the integral of the total energy $T+U$.

Also, the statement is true for any "generalized" coordinates q. and leads to

1) Lagrangian mechanics

2) Hamiltonian mechanics

Ex Taut string (Least action continuum mech)



Kinetic Energy

$$T = \frac{1}{2}\int_0^L \rho\, u_t^2\, dx \qquad \rho = \text{mass/length}$$

Potential Energy

$$V = \int_0^L \tau(x,t)\left(\sqrt{1+u_x^2} - 1\right) dx$$

where $\tau(x,t)$ = local tensile force.
Here V is the work done to stretch the string (only). Note $V = 0$ if $u_x \equiv 0$.

Least Action

$$J(u) = \int_0^t \int_0^L (T - V)\, dx\, dt$$

extremize over $u(x,t)$ satisfying

$$u(0,t) = u(L,t) = 0 \qquad \text{B.C.}$$

$$u(x,0),\ u_t(x,0) \text{ specified}$$