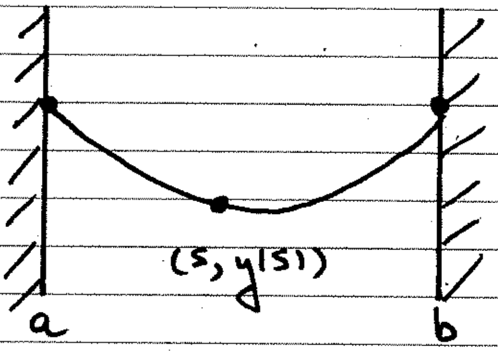
Isoperimetric constraints - Hanging cable



s = arc length

Cable of length L has a shape which minimizes the gravitational potential energy.

Potential Energy μ = weight/length

$$J(y) = \mu \int_0^L y(s)\, ds$$

Fixed Span

$$K(y) = \int_0^L dx(s) = \int_0^L \sqrt{1 - y'(s)^2}\, ds = b - a = K_0$$

Latter follows from unit tangent:

$$\hat{T}(s) = (x'(s), y'(s))$$

$$\| \hat{T} \| = x'(s)^2 + y'(s)^2 = 1$$

Thus the hanging cable problem amounts to minimizing $J(y)$ subject to the constraint $K(y) = K_0$ for y with $y(a) = y(b) = 0$.

Alternately

(1) $\mathcal{A} \equiv \{ y \in C^2[a,b] : y(a) = y(b) = 0, \; K(y) = K_0 \}$

where $K_0 = b - a$. Then seek minimer $y \in \mathcal{A}$ of

(2) $$\min_{y \in \mathcal{A}} J(y)$$

Note that the set

$$\mathcal{A}^{**} = \{ h \in C^2[a,b] : \; h(a) = h(b) = 0 \}$$

is <u>not</u> the set of admissible variations!

While it is true that

$$y \in \mathcal{A}, \; h \in \mathcal{A}^{**} \quad \Rightarrow \quad y + \varepsilon h \in \mathcal{A}^{**}$$

it is not true $y + \varepsilon h \in \mathcal{A}$ since, in particular, $y + \varepsilon h$ need not satisfy the integral constraint

Short hand notation for (1)-(2) is

$$\min_{y \in \mathcal{A}^{**}} J(y) \quad \text{sub. to. constraint } K(y) = K_0$$

Isoperimetric Problems

$$J(y) \equiv \int_a^b F(x, y, y')\,dx \qquad \text{object}$$

$$K(y) \equiv \int_a^b G(x, y, y')\,dx \qquad \text{constraint}$$

Admissible set

$$\mathcal{A} = \left\{ y \in C^2[a,b] \; : \; y(a) = A,\ y(b) = B,\ K(y) = K_0 \right\}$$

where K_0 is some constant. Seek to minimize $J(y)$ subject to $K(y) = K_0$

Note

(1) $$\min_{y \in \mathcal{A}} J(y) = \min_{y \in \mathcal{A}} \Big(J(y) + \lambda \big(\underbrace{K(y) - K_0}_{0 \text{ if } y \in \mathcal{A}} \big) \Big)$$

for any $\lambda \in \mathbb{R}$ (Lagrange multipliers).

Modify Lagrangian:

$$L(y) \equiv J(y) + \lambda K(y)$$

$$L(y) = \int_a^b \underbrace{(F + \lambda G)}_{H}\,dx$$

Thus we seek to extremize $L(y)$ over A.

Toward this end we again define

$$A^{**} = \{ h \in C^2[a,b] : h(a) = h(b) = 0 \}$$

noting $h \in A^{**}$ need not be admissible variations.

Let $\bar{y} \in A$ be an extrema. Define

$$(1) \qquad I(\varepsilon_1, \varepsilon_2) \equiv L(\bar{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2)$$

where $\varepsilon_i \in \mathbb{R}$ and $h_j(x) \in A^{**}$. Define

$$(2) \qquad y \equiv \bar{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2$$

Again y need not be in A. However

$$(3) \qquad K(y) = f(\varepsilon_1, \varepsilon_2) = K_0$$

for any choice of h_i implies that there is a functional dependence on $\varepsilon_1, \varepsilon_2$ such that $y \in A$!

Lastly, if $\bar{y}$ is the extrema it necessarily occurs at

$$(\varepsilon_1, \varepsilon_2) = (0,0)$$

Key fact

$\bar{y} \in A$ extrema $\Rightarrow$ $I(\varepsilon_1, \varepsilon_2)$ has extrema at $(0,0)$ $\forall \eta_i \in A^{**}$

This is the necessary condition required to derive the problem $\bar{y}$ must be a solution of.

$$I(\varepsilon_1, \varepsilon_2) = \int_a^b H(x, y, y', \lambda)\, dx$$

where $H = F + \lambda G$. Necessarily

$$\left.\frac{\partial I}{\partial \varepsilon_i}\right|_{(0,0)} = \int_a^b \underbrace{\left(H_y - \frac{d}{dx} H_{y'}\right)}_{0} h_i(x)\, dx$$

for all $h_i(x) \in A^{**}$. Conclude $\bar{y} \in A$ is a solution of

$$(1) \quad F_y + \lambda G_y = \frac{d}{dx}\left(F_{y'} + \lambda G_{y'}\right)$$

$$(2) \quad y(a) = A \qquad y(b) = B$$

$$(3) \quad K(y) = K_0$$

Eqns (1)-(2) define a family of functions $y_\lambda(x)$. Problem is completely solved by find λ s.t.

$$K(y_\lambda) = K_0$$

Alternate derivation / remarks

Recall for extrema $\bar{y}$ we defined

$$y = \bar{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2$$

and $y \in A$ if

(1) $$K(y) = f(\varepsilon_1, \varepsilon_2) = K_0$$

Implicit function theorem (can) imply

$$\varepsilon_2 = \phi(\varepsilon_1)$$

for some ϕ. Hence

$$y = \bar{y} + \varepsilon_1 h_1 + \phi(\varepsilon_1) h_2 \in A \qquad !$$

for all ε_1 near $\varepsilon_1 = 0$. Then for

$$I(\varepsilon_1) \equiv L(y)$$

we must have (after calculations)

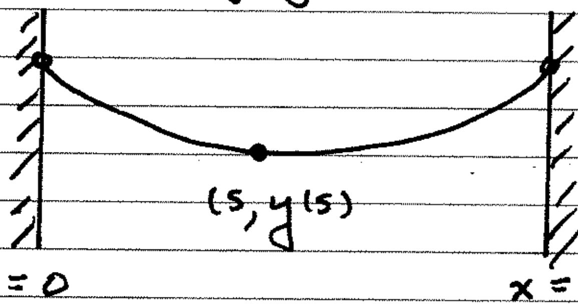
$$\left.\frac{dI}{d\varepsilon_1}\right|_{\varepsilon_1=0} = \int_a^b \left(H_y - \frac{d}{dx} H_{y'}\right)\left(h_1 + \phi'(0) h_2\right) dx = 0$$

$\forall h_i \in A^{**}$. We then, again, conclude

$$H_y = \frac{d}{dx} H_{y'}$$

with $\bar{y} \in A$.

EXAMPLE **Hanging cable**



$y(0) = 0$

$y(1) = 0$

$$J(y) = \int_0^{\ell} \mu\, y(s)\, ds \qquad \text{P.E.}$$

$$K(y) = \int_0^{\ell} \sqrt{1 - y'(s)^2}\, ds$$

Here the cable length is ℓ. Must extremize

$$L(y) = \int_0^{\ell} \underbrace{\left(\mu\, y(s) + \lambda \sqrt{1 + y'(s)^2}\right)}_{H} ds \qquad \text{modified Lagrangian}$$

Here λ is a Lagrange multiplier. EL-eqn

$$H_y = \frac{d}{ds} H_{y'}$$

yields

$$(1) \qquad \hat{\mu} = \frac{d}{ds}\left(\frac{y'(s)}{\sqrt{1 - y'(s)^2}} \right), \qquad \hat{\mu} = \frac{\mu}{\lambda}$$

Integrate eqn (1) in s and solve for $y'(s)^2$

$$y'(s)^2 = \frac{(s+c)^2}{\hat{\lambda}^2 + (s+c)^2} \qquad \hat{\lambda} \equiv \frac{\lambda}{\mu}$$

and $c \in \mathbb{R}$ const. of integration.

Since $y(0) = 0$ we have

$$y(s) = \int_0^s \frac{(t+c)}{\sqrt{\hat{\lambda}^2 + (t+c)^2}}\, dt$$

Evaluating

$$y(s) = \sqrt{\hat{\lambda}^2 + (t+c)^2}\,\Big|_0^s$$

Using $y(\ell) = 0$ we find $c = \frac{1}{2}\ell$. Hence

$$\boxed{y_\lambda(s) = \sqrt{\hat{\lambda}^2 + (s - \ell/2)^2} - \sqrt{\hat{\lambda}^2 + \tfrac{1}{4}\ell^2}}$$

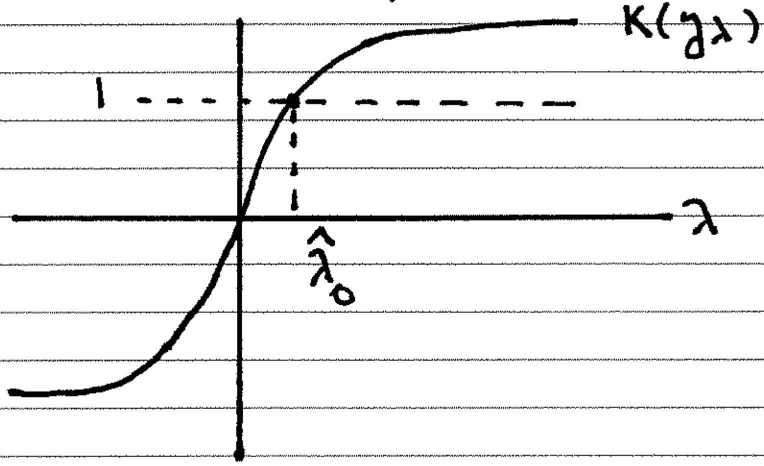
One finds $\hat{\lambda}$ from the span constraint.

$$(1) \quad K(y_\lambda) = \int_0^\ell \sqrt{1 - y_\lambda'(s)^2}\, ds = 1$$

After some calculations

$$K(y_\lambda) = \lambda \ln \left(\frac{L + \sqrt{4\lambda^2 + L^2}}{-L + \sqrt{4\lambda^2 + L^2}} \right)$$

which can be plotted



Then $\hat{\lambda}_0$ used in $y_{\hat{\lambda}}(s)$ <u>is</u> the soln of the isoperimetric problem.