Principle of Least Action

Let $q \in \mathbb{R}^n$ be a coordinate system for some physical system and:

$$T(q,\dot{q}) = \text{kinetic energy}$$

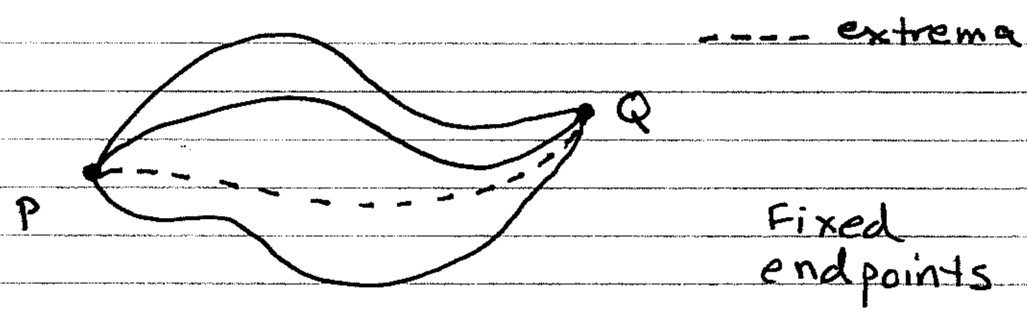
$$V(q,\dot{q}) = \text{potential energy}$$

More generally V may depend on time t.

$$L \equiv T - V \qquad \text{Lagrangian}$$

Then the path followed by a (holonomic) system on $[t_1, t_2]$ is that which extremizes the <u>action</u>

$$J(q) \equiv \int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$$



Conjugate Momenta and Generalized Force

For $q = (q_1, \ldots q_n)$ the EL-eqns which extremize the action are

$$(1) \qquad \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

For each coordinate q_i we define a conjugate momenta p_i and generalized force F_i by

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

$$F_i \equiv \frac{\partial L}{\partial q_i}$$

Then the equations of motion (1) become

$$\frac{d\vec{p}}{dt} = \vec{F}$$

which is very similar to Newtons Law.

Common Kinetic Energy Coordinates

EX Cartesian $q = (x, y, z)$

$$T = \frac{1}{2} m \dot{q}^T \dot{q} = \frac{1}{2} m \dot{q}_i^2 \qquad \text{(sum)}$$

EX Cylindrical $q = (r, \theta, z)$

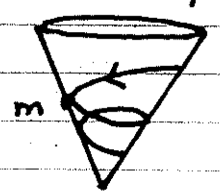
$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)$$

EX Spherical $q = (\rho, \phi, \theta)$

$$T = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \sin^2\phi \ \dot{\theta}^2)$$

Lagrangian Mechanics

Often systems have constraints (holonomic) which functionally related coordinates



$\phi(q) = z^2 - x^2 - y^2 = 0$
on cone

So a more general class of problems in Lagrangian Mechanis is

$$\boxed{\begin{array}{ll} \dfrac{dp_i}{dt} = \dfrac{\partial L}{\partial q_i} & i = 1, \ldots n \\[2ex] \phi_j(q, t) = 0 & j = 1, \ldots m < n \end{array}}$$

Latter are "holonomic" constraints.

EXAMPLE Planar motion with central potential

Let (r, θ) be polar coordinates and define

$$L(q, \dot{q}) = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$$

Since $L_\theta = 0$ the momenta $p_\theta \equiv L_{\dot{\theta}} = mr^2\dot{\theta}$ conjugate to θ is conserved:

$$\frac{dp_\theta}{dt} = 0 \qquad\qquad p_\theta = mr^2\dot{\theta}$$

This is conservation of angular momentum!

The second EL-eqn $L_r = \frac{d}{dt} L_{\dot{r}}$ implies (then)

$$(1) \qquad m\ddot{r} = \frac{p_\theta^2}{mr^3} - U'(r)$$

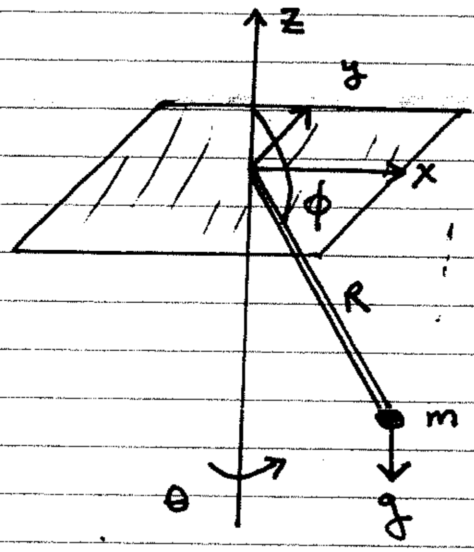
Solving the equations of motion reduces to the integrability of (1) of the general for

$$\ddot{r} + G(r) = 0$$

Given (1), one can then prove energy E is also conserved

$$\frac{dE}{dt} = \frac{d}{dt}(T+U) = \left(m\ddot{r} - \frac{p_\theta^2}{mr^3} + U'(q)\right)\dot{r} = 0$$

EXAMPLE Spherical Pendulum (length R)



Holonomic constraint

$$x^2 + y^2 + z^2 = R^2$$

In spherical (ρ, ϕ, θ) is

$$\rho = R$$

Transformations:

$$x = R\cos\theta\sin\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\phi$$

Kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ using $\dot{R} = \dot{\rho} = 0$

$$T = \frac{1}{2}m\left(R^2\dot{\theta}^2\sin^2\phi + R^2\dot{\phi}^2\right)$$

Potential energy (gravity) is $V = mgz$ or

$$V = mgR\cos\phi$$

Generalized coordinates $q = (q_1, q_2) = (\theta, \phi)$

Since $L \equiv T - V$ and $L_\theta = 0$ its associated conjugate momentum p_θ is constant

$$(1) \qquad \boxed{p_\theta = L_{\dot{\theta}} = mR^2\dot{\theta}\sin^2\phi}$$

Conjugate momentum p_ϕ is not constant.

Given

$$L_{\dot\phi} = p_\phi = mR^2\dot\phi$$

(2) $$L_\phi = mR^2\dot\theta^2 \sin\phi\cos\phi - mgR\sin\phi$$

The EL-eqns are (for ϕ)

(3) $$mR^2\ddot\phi = L_\phi$$

Conservation law (1) can be used to eliminate $\dot\theta$ indicated in (2). Ultimately (3) reduces to

(4) $$\ddot\phi + G(\phi) = 0$$

where

$$-G(\phi) = \left(\frac{p_\theta^2}{m^2R^4}\right)\frac{\cos\phi}{\sin^3\phi} + \frac{g}{R}\sin\phi$$

The system (4) is Hamiltonian (Dyn Sys Theory) and in principal integrable:

$$\dot\phi\ddot\phi + G(\phi)\dot\phi = 0$$

(5) $$\frac{1}{2}\dot\phi^2 + V(\phi) = c_1 \in \mathbb{R}$$

for the (new) potential

$$V(\phi) = \int^\phi G(s)\,ds.$$

Solving (5) for $\dot\phi$ yields a separable nonlinear first order problem!!