__Multiple dependent variables__ $y = (y_1, \dots, y_n)$

$$J(y) = \int_{t_1}^{t_2} L(t, y, y')\, dt$$

A set of admissible functions could be

$$\mathcal{A} = \{ y : y_i \in C^2[t_1, t_2],\ y_i(t_1) = \alpha_i,\ y_i(t_2) = \beta_i \}$$

and associated admissible variations

$$\mathcal{A}^* = \{ h : h_i \in C^2[t_1, t_2],\ h(t_1) = h(t_2) = \vec{0} \}$$

Variation is computed in same manner

$$F(\varepsilon) \equiv J(y + \varepsilon h)$$

First variation is $F'(0)$ which is

$$(1) \quad \delta J = \underbrace{\cancel{L_{y_i'} h_i \Big|_{t_1}^{t_2}}}_{h \in \mathcal{A}^*} + \int_{t_1}^{t_2} \left(L_{y_i} - \frac{d}{dt} L_{y_i'} \right) h_i \, dx$$

where we have adopted the Einstein repeated index sum convention. That is repeated indices in products imply sums:

$$x_i y_i = x_1 y_1 + \cdot \cdot x_n y_n$$

Hence (1) implies a set of EL-Eqns

$$\boxed{L_{y_i} = \frac{d}{dt} L_{y_i'}} \qquad i = 1 \dots n$$

EXAMPLE $J(y) = \int_0^1 (y_1^2 + y_1' y_2' + 12 y_2)\, dt$

We assume $y_1(0) = y_1(1) = y_2(1) = 0$ and $y_2(0) = 1$.
The EL-eqn's are

$$L_{y_1} = \frac{d}{dt} L_{y_1'}$$
$$L_{y_2} = \frac{d}{dt} L_{y_2'}$$

For the given Lagrangian

$$(1) \qquad 2y_1 = y_2''$$
$$(2) \qquad 12 = y_1''$$

Fourth order linear system. Eqn (2) implies

$$y_1 = 6t^2 + c_1 t + c_2$$

Use $y_1(0) = y_1(1) = 0$ to get

$$y_1(t) = 6t^2 - 6t$$

Since y_1 known, Eqn (1) with B. Cond on y_2 give $y_2(t)$

$$y_2(t) = t^4 - 2t^3 + 1$$

Geodesics in $\mathbb{R}^3$

Let surface S have parametrization

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

Of all paths Γ on S connecting P_1 and P_2 on S, which has the shortest length? Such paths are called geodesics.

Parametrize path $r(t) = (X(t), Y(t), Z(t))$

$$X(t) = x(u(t), v(t))$$
$$Y(t) = y(u(t), v(t))$$
$$Z(t) = z(u(t), v(t))$$

Arclength

$$L = \int_{t_1}^{t_2} (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)^{1/2}\, dt \tag{1}$$

where

$$\dot{X} = x_u \dot{u} + x_v \dot{v} \tag{2}$$

$$\dot{Y} = y_u \dot{u} + y_v \dot{v} \tag{3}$$

$$\dot{Z} = z_u \dot{u} + z_v \dot{v} \tag{4}$$

Since $\dot{X}, \dot{Y}, \dot{Z}$ are functions of $(u, v, \dot{u}, \dot{v})$ the integrand is as well.

Recall $\|\dot{r}\| = \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}$. Using (2)–(3)

(4) $$\|\dot{r}\| = \sqrt{P\dot{u}^2 + 2Q\dot{u}\dot{v} + R\dot{v}^2} = L(u, v, \dot{u}, \dot{v})$$

where

$$P = x_u^2 + y_u^2 + z_u^2$$

$$Q = x_u x_v + y_u y_v + z_u z_v$$

$$R = x_v^2 + y_v^2 + z_v^2$$

In summary

$$J(u,v) = \int_{t_1}^{t_2} L(u, v, \dot{u}, \dot{v})\, dt$$

must be minimized. Here the Lagrangian depends on two functions and the EL-eqns are

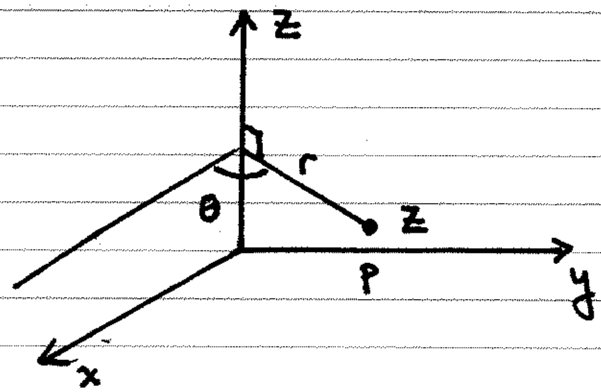
$$L_u = \frac{d}{dt} L_{\dot{u}}$$

$$L_v = \frac{d}{dt} L_{\dot{v}}$$

The B.C. are $(u(t_k), v(t_k)) = (U_k, V_k)$ at $k = 1, 2$. Once EL eqns are solved with these B.C. the path coordinates can be reconstructed from $X(t) = x(u(t), v(t)), \ldots$

EXAMPLE Cylindrical Coordinates

Suppose a surface can be described by a graph $z = f(x,y)$.



In polar coordinates $(u,v) = (r,\theta)$ and the surface parametrization is

$$x = x(r,\theta) = r\cos\theta$$

$$y = y(r,\theta) = r\sin\theta$$

$$z = z(r,\theta) = f(r\cos\theta, r\sin\theta)$$

An explicit example would be

$$z = f(x,y) = x^2 + y^2 + x$$

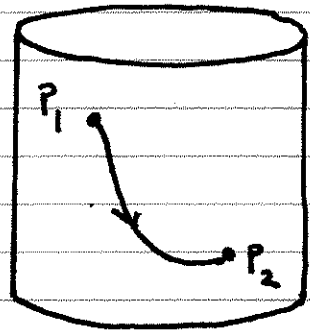
in which case

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = r^2 + r\cos\theta$$

EXAMPLE Geodesics on a cylinder



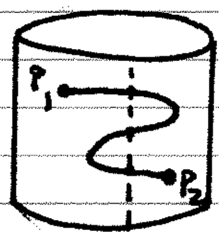
Use polar coordinates

$$x = a\cos\theta$$

$$y = a\sin\theta$$

$$z = z(\theta)$$

where a is the radius. We have tacitly assumed that z is a function of θ which need not be the case as the following figure illustrates.



3 different z values for same θ

Under this simplifying reduction

$$r(\theta) = (a\cos\theta, a\sin\theta, z(\theta))$$

$$r'(\theta) = (-a\sin\theta, a\cos\theta, z'(\theta))$$

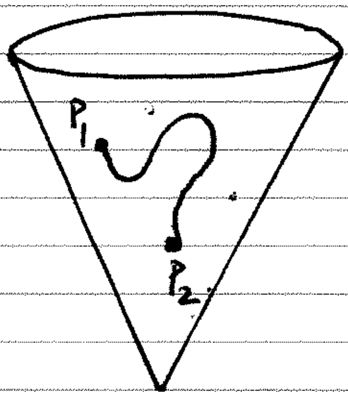
Computing $\|r'(\theta)\|$ we find the arclength functional

$$J(z) = \int_{\theta_1}^{\theta_2} \sqrt{a^2 + z'(\theta)^2}\, d\theta$$

$$\mathcal{A} = \{ z \in C^2[\theta_1, \theta_2] : z(\theta_k) = z_k \}$$

The solution has $z(\theta)$ linear in θ, as expected.

EXAMPLE Geodesics on a cone $x^2 + y^2 = a^2 z^2$



The surface in polar is parametrized by

$$x = x(\theta, r) = r\cos\theta$$

$$y = y(\theta, r) = r\sin\theta$$

$$z = z(\theta, r) = ar$$

For the path above neith r nor z are functions of θ. We shall first formulate the problem for this case and then assume $r = r(\theta)$, an intuitive assumption.

Let $r = r(t)$, $\theta = \theta(t)$. Then

$$\dot{X} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$\dot{Y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$
$$\dot{Z} = a\dot{r}$$

The Lagrangian $L = \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}$ then simplifies

$$J(r, \theta) = \int_{t_1}^{t_2} \sqrt{r^2\dot{\theta}^2 + (1+a^2)\dot{r}^2}\, dt$$

The resulting EL-eqns are difficult to solve but they have one first integral since

$$L_\theta = 0$$

If we now assume $r = r(\theta)$ on extrema

$$x(\theta) = r(\theta)\cos(\theta)$$

$$y(\theta) = r(\theta)\sin(\theta)$$

$$z(\theta) = a\, r(\theta)$$

The arclength Lagrangian is, now,

$$L = \sqrt{(x')^2 + (y')^2 + (z')^2} \qquad (\ )' = \frac{d}{d\theta}(\ )$$

Simplifying this

$$J(r) = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (1+a^2)(r')^2}\, d\theta$$

$$\mathcal{A} = \{ r \in C^2[\theta_1, \theta_2] : r(\theta_k) = r_k \}$$

Solving the Euler Lagrange eqns,

Since $L_\theta = 0$ we have the first integral

$$(1) \quad L - r' L_{r'} = \sqrt{r^2 + b^2 r'^2} - \frac{r'^2 b^2}{\sqrt{r^2 + b^2 r'^2}} = k$$

for some constant k.

One method is to solve

$$L - r' L_{r'} = k$$

for r' to get

$$\frac{dr}{d\theta} = \frac{r\sqrt{r^2 - k^2}}{kb}$$

which is separable. The resulting integrals are doable but very messy.

A rather brilliant solution in some books involves the observation that the term $\sqrt{r^2 + b^2 r'^2}$ looks like an inverse trig sub. Letting

$$(2) \qquad r = c_1 \sec\left(\frac{\theta}{b} + c_2\right)$$

and using $\tan^2 x + 1 = \sec^2 x$ and (2) in (1) we find (1) is satisfied $\forall \theta$ if $k = c$.

Therefore (2) is the solution. c_K must be chosen to satisfy B.C.