Several Independent Variables

(1) $$J(u) \equiv \int_\Omega L(x, u, \nabla u)\, dx \qquad \Omega \subset \mathbb{R}^3$$

over

$$\mathcal{A} \equiv \{ u \in C^1(\bar{\Omega}) : \ u|_{\partial\Omega} = \gamma \}$$
$$\mathcal{A}^* = \{ \eta \in C^1(\bar{\Omega}) : \ \eta|_{\partial\Omega} = 0 \}$$

Here $u = u(x)$ where $x = (x_1, x_2, x_3)$ and we adopt the notation

$$\nabla_{u'} L = \left(\frac{\partial L}{\partial z_1}, \frac{\partial L}{\partial z_2}, \frac{\partial L}{\partial z_3} \right)$$

where $L = L(x, u, z)$ and $z = (z_1, z_2, z_3)$.

We seek to compute the first variation

$$\delta J(\bar{u}, \eta) = \frac{d}{d\varepsilon} J(\bar{u} + \varepsilon\eta) \Big|_{\varepsilon = 0}$$

Given (1)

(2) $$\delta J(\bar{u}, \eta) = \int_\Omega (L_u \eta + \underbrace{\nabla_{u'} L \cdot \nabla\eta})\, dx$$

Need to integrate by parts (indicated term)

Use the following identity

$$\nabla \cdot (a\vec{b}) = \nabla a \cdot \vec{b} + a\, \nabla \cdot \vec{b}$$

where $a = \eta$, $\vec{b} = \nabla_{u'} L$. Hence

$$(3) \quad \nabla \cdot (\eta \nabla_{u'} L) = \underbrace{\nabla \eta \cdot \nabla_{u'} L} + \eta\, \nabla \cdot \nabla_{u'} L$$

Noting indicated terms in (2)-(3) are the same, the variation δJ in (2) becomes:

$$(4) \quad \delta J = \int_{\Omega} \Big(L_u \eta + \nabla \cdot (\underset{\Uparrow}{\eta} \nabla_{u'} L) - \eta\, \nabla \cdot \nabla_{u'} L \Big)\, dx$$

Divergence Theorem applied to $\Uparrow$ term (int. by parts) gives

$$\delta J = \underbrace{\int_{\partial\Omega} \eta \nabla_{u'} L \cdot \hat{N}\, ds}_{\substack{\text{vanishes since} \\ \eta = 0 \text{ on } \partial\Omega}} + \int_{\Omega} \underbrace{(L_u - \nabla \cdot \nabla_{u'} L)}_{=0} \eta\, dx$$

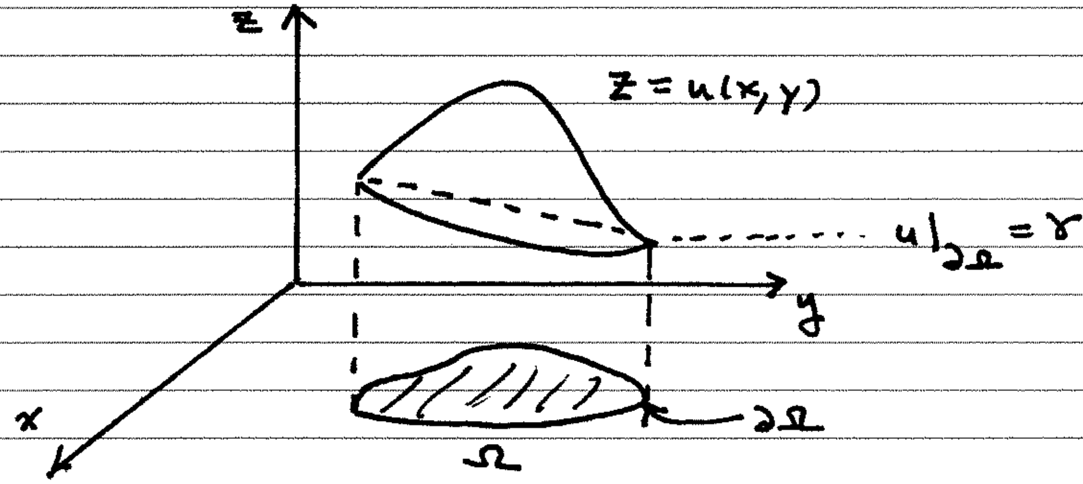
Euler Lagrange Eqn is

$$\boxed{L_u = \nabla \cdot \nabla_{u'} L}$$

Minimum Area Problem

$$J(u) = \int_{\Omega} L(\nabla u)\, dx = \int_{\Omega} \sqrt{1 + u_x^2 + u_y^2}\, dA$$

for $u \in \mathcal{A}$ is the surface area (below) of the graph $z = u(x,y)$ over $\Omega \subset \mathbb{R}^2$ with boundary height γ on $\partial\Omega$.



Note $L = L(z_1, z_2) = \sqrt{1 + z_1^2 + z_2^2}$ in the alternate notations. Making this idenification

$$\nabla_{u'} L = \left(\frac{\partial L}{\partial u_x}, \frac{\partial L}{\partial u_y} \right)$$

$$\nabla_{u'} L = \left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}, \frac{u_y}{\sqrt{1+u_x^2+u_y^2}} \right)$$

Hence the EL-eqn

$$L_u = \nabla \cdot \nabla_{u'} L$$

becomes

$$\frac{\partial}{\partial x}\left(\frac{u_x}{L}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{L}\right) = 0$$

Expanding out

$$(1) \quad \boxed{u_{xx}(1+u_y^2) - 2u_x u_y u_{xy} + u_{yy}(1+u_x^2) = 0} \quad \text{(EL)}$$

Highly nonlinear. Some special solns include

<u>CASE 1</u> $\gamma = \gamma_0$ constant on $\partial\Omega$. Then $u(x,y) \equiv \gamma_0$ is in A and solves (1). This is the flat plane soln

<u>CASE 2</u> $\gamma = \varepsilon \beta(x,y)$ on $\partial\Omega$ where $0 < \varepsilon \ll 1$. Here boundary curve has a small amplitude. Use

$$u(x,y) = \varepsilon u_0(x,y) + \varepsilon^2 u_1(x,y) + O(\varepsilon^3)$$

Substitute into (1) and retain $O(\varepsilon)$ terms

$$(2) \qquad \nabla^2 u_0 = 0 \qquad (x,y) \in \Omega$$

$$(3) \qquad u_0 = \beta \qquad (x,y) \in \partial\Omega$$

is a Laplacian B.V.P.

Self Adjoint Elliptic PDEs

(1) $$J(u) \equiv \int_\Omega \underbrace{(p(\nabla u)^2 + qu^2)}_{L(x,u,\nabla u)} dx + \int_{\partial\Omega} \underbrace{(\frac{\alpha}{\beta} p u^2)}_{G(x,u)} dS$$

with the understanding $(\nabla u)^2 = \nabla u \cdot \nabla u$. Here

$$\mathcal{A} = \{ u \in C^2(\bar{\Omega}) : \ \alpha u + \beta \frac{\partial u}{\partial n} = \underline{0} \ , x \in \partial\Omega \}$$

hence $\mathcal{A} = \mathcal{A}^*$. Note $\frac{\partial u}{\partial n} = \nabla u \cdot \hat{N}$ where $\hat{N}$ is a unit outward normal to Ω.

As before (with new added boundary term)

$$\delta J = \int_\Omega (L_u \eta + \underbrace{\nabla \eta \cdot \nabla_{u'} L}) dx + \int_{\partial\Omega} G_u \eta \, dS$$

where $u = \bar{u} + \varepsilon\eta$. Integrate by parts

(2) $$\delta J = \int_\Omega \underbrace{(L_u - \nabla \cdot \nabla_{u'} L)}_{EL} \eta \, dx + \int_{\partial\Omega} \underbrace{(G_u + \nabla_{u'} L \cdot \hat{N})}_{BC} \eta \, dS$$

This must vanish for all $\eta \in \mathcal{A}^*$ where in particular $\eta \neq 0$ on $\partial\Omega$.

Consider the boundary integrand BC first

$$G_u + \nabla_{u'} L \cdot \hat{N} = \left(\frac{2\alpha p}{\beta}\right) u + 2p\,(u_x, u_y, u_z) \cdot \hat{N}$$

$$= \frac{2\alpha p}{\beta} u + 2p \frac{\partial u}{\partial n}$$

$$= \frac{2p}{\beta} \underbrace{\left(\alpha u + \beta \frac{\partial u}{\partial n} \right)}_{0 \text{ since } u \in A.}$$

Despite $\eta \neq 0$ on $\partial\Omega$ the boundary integral vanishes hence

$$\boxed{L_u - \nabla \cdot \nabla_{u'} L = 0} \qquad \text{(EL)}$$

Again, noting

$$L_u = 2qu \qquad \nabla_{u'} L = 2p\,(u_x, u_y, u_z)$$

we ultimately find

$$\boxed{\begin{aligned} -\nabla \cdot (p \nabla u) + qu &= 0 \qquad x \in \Omega \\ \alpha u + \beta \frac{\partial u}{\partial n} &= 0 \qquad x \in \partial\Omega. \end{aligned}}$$

This is the PDE Sturm Liouville problem. The solution can be numerically approximated by attempting to minimize $J(u)$ iteratively.

Constrained Problems

Seek to extremize

$$J(u) = \int_\Omega |\nabla u|^2 dx = \int_\Omega (u_x^2 + u_y^2 + u_z^2)\, d\vec{x}$$

over u that vanish on $\partial\Omega$. We further require

$$K(u) = \int_\Omega r u^2 dx = 1 \qquad r > 0$$

which is a normalization constraint with respect to the inner product

$$\langle u, v \rangle = \int_\Omega r u v\, dx$$

The admissible set is, therefore

$$A = \{ u \in C^2(\bar{\Omega}) : u|_{\partial\Omega} = 0, \ K(u) = 1 \}$$

As with one dimensional problems one uses a Lagrange multiplier λ and extremize

$$M(u) = J(u) + \lambda K(u)$$

(1)
$$M(u) = \int_\Omega L(u, \nabla u, \lambda)\, dx$$

The EL-eqns are (again)

$$(2) \qquad L_u = \nabla \cdot \nabla_{u'} L$$

where the modified Lagrangian is

$$(3) \qquad L = u_x^2 + u_y^2 + u_z^2 + \lambda r u^2$$

Given $L_u = 0$ and $\nabla_{u'} L = 2(u_x, u_y, u_z) = 2\nabla u$ the (EL) equations are

$$\boxed{\begin{array}{rcll} \nabla^2 u + \lambda r u & = & 0 & x \in \Omega \\ u & = & 0 & x \in \partial\Omega \\ \|u\| & = & 1 & \end{array}}$$

Here the norm is w.r.t. $\langle u, v \rangle = \int_\Omega r u v \, dx$.

This is an eigenvalue problem for eigenvalue λ and $\|u\| = 1$ is required to narrow down the nomalized eigenfunction.

Convexity Example

Minimum surface area functional

$$J(u) = \int_\Omega L(\nabla u)\, dx = \int_\Omega \sqrt{1+|\nabla u|^2}\, dx$$

we shall use the notation

$$L(X, Y) = \sqrt{1 + X^2 + Y^2} \qquad P = (X, Y)$$

so that

$$L(\nabla u) = \sqrt{1 + u_x^2 + u_y^2}$$

By definition the functional $J(u)$ is convex if

$$(1) \qquad J(u+\eta) - J(u) - \delta J(u, \eta) \geq 0$$

for all $u \in A$, $\eta \in A^*$. For our Lagrangian

$$\delta J = \int_\Omega (L_X \eta_x + L_Y \eta_y)\, dA$$

Thus a sufficient (pointwise) condition for functional convexity is

$$(2) \qquad L(P+\Delta P) - L(P) - \nabla_P L \bullet \Delta P \geq 0$$

where L_i evaluated at P with

$$P = (u_x, u_y) \qquad \Delta P = (\eta_x, \eta_y)$$

Alternate notation would be (long hand)

$$L(u_x+\eta_x, u_y+\eta_y) - L(u_x, u_y) - L_X(u_x,u_y)\eta_x - L_Y(u_x,u_y)\eta_y \geq 0$$

Stated more succintly, if the map

$$(X, Y) \longmapsto L(X, Y)$$

$$P \longmapsto L(P)$$

is convex then (2) is satisfied for all u, η and thefore (1) is true, i.e. $J(u)$ is a convex functional.

The map $(X,Y) \longmapsto L(X,Y)$ is convex if its associated Hessian is positive definite.

$$H = \begin{bmatrix} L_{XX} & L_{XY} \\ L_{YX} & L_{YY} \end{bmatrix} = \delta \begin{bmatrix} 1+Y^2 & -X \\ -XY & 1+X^2 \end{bmatrix}$$

where $\delta = (1+X^2+Y^2)^{-2}$. From this

$$\det H = (1+X^2+Y^2)^{-2} > 0$$

$$L_{XX} = \delta^{-1}(1+Y^2) > 0$$

implies H (symmetric) has (strictly) positive eigenvalues and pos. def.

<u>Conclude:</u> if $\exists \bar{u} \in A$ s.t $\delta J(\bar{u}, \eta) = 0 \; \forall \eta \in A^*$ then $\bar{u}$ is a <u>minimizer</u>