Free endpoint (Transversal) Conditions

Let a curve C be defined by $f(x,y)=0$ and define

$$J(y,t) \equiv \int_a^t L(x, y(x), y'(x))\, dx$$

Seek to extremize J over

$$A(t) \equiv \{ y \in C^1[a,t] : y(a) = \alpha,\ f(t, y(t)) = 0 \}$$

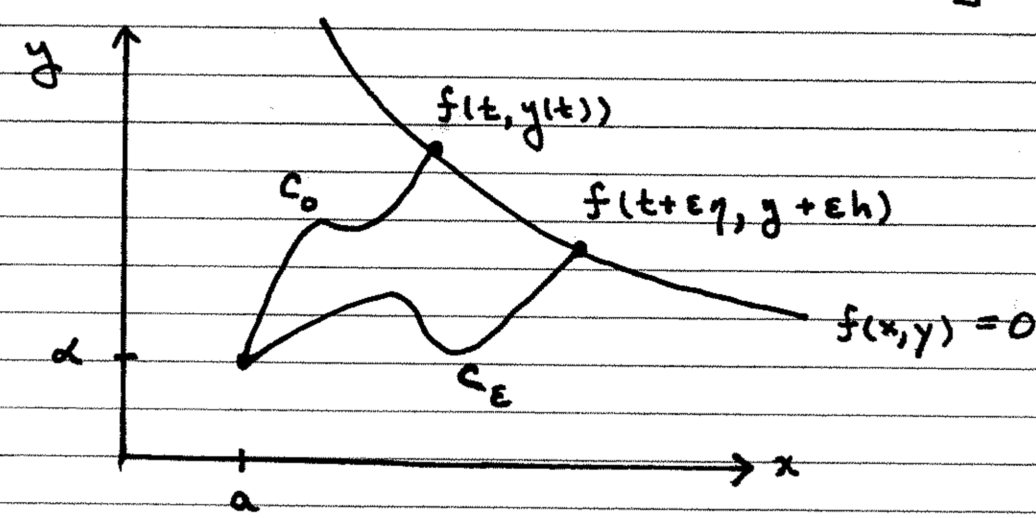
for some range of t. Note that as a functional

$$J : C^1[a,b] \times \mathbb{R} \to \mathbb{R}$$

with the norm

$$\| (y,t) \| \equiv \| y \|_p + |t|$$

Schematic of variations ($\alpha = 0$ wlog)



Use Lagrange multiplier method.

$$I(y(t),t) \equiv \int_a^t L(x,y(x),y'(x))\,dx + \lambda f(t,y(t))$$

Suppose $(y(t),t)$ is the admissible solution. We consider local variations

$$z_\varepsilon = (y(t)+\varepsilon h(t),\ t+\varepsilon\eta)$$

where $h(t)$ and η are fixed. λ is the Lagrange multiplier. Define

$$F(\varepsilon) = I(z_\varepsilon)$$

A necessary condition is given (as usual):

$$F'(0) = \delta I = 0 \qquad \forall h,\eta$$

Explicitly

$$F(\varepsilon) = \int_a^{t+\varepsilon\eta} L(x,y(x)+\varepsilon h(x),y'(x)+\varepsilon h'(x))\,dx$$

$$+ \lambda f(t+\varepsilon\eta,\ \underbrace{y(t+\varepsilon\eta)+\varepsilon h(t+\varepsilon\eta)}_{\text{critical point.}})$$

Make use of Leibnitz rule. An example

$$\frac{d}{d\varepsilon}\int_a^{b(\varepsilon)} g(x,\varepsilon)\,dx = g(b(\varepsilon),\varepsilon)\,b'(\varepsilon) + \int_a^{b(\varepsilon)} g_\varepsilon(x,\varepsilon)\,dx$$

After considerable calculations (at $\varepsilon = 0$)

$$\delta I = L\eta + \lambda\left(f_t\eta + f_y\,[y'\eta + h]\right) + \int_a^t (L_y h + L_{y'} h')\,dx$$

Integrate by parts and collect terms

$$\delta I = \underbrace{[L + \lambda(f_t + f_y y')]\eta}_{\text{at } x=t} + \underbrace{[L_{y'} + \lambda f_y]h}_{\text{at } x=t} + \int_a^t \underbrace{(L_y - \tfrac{d}{dx}L_{y'})}_{EL} h\,dx$$

Hence extrema satisfy

(1) $\quad L_{y'} + \lambda f_y = 0 \qquad$ at $x=t$

(2) $\quad L + \lambda(f_t + f_y y') = 0 \qquad$ at $x=t$

Use (1)-(2) to eliminate λ and obtain the <u>transversality condition</u>

$$\boxed{L f_y = L_{y'}(f_t + f_y y'), \quad x=t}$$

Solution is found:

(3) $L_y = \frac{d}{dx} L_{y'}$ Euler-Lagrange

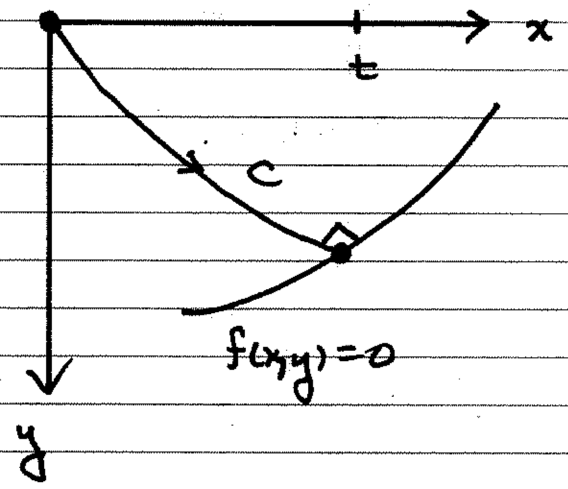
(4) $y(a) = \alpha$ Given B.C.

(5) $L f_y = L_{y'} (f_t + f_y y')$ Transversality.

<u>Remark</u> Note that if $f_y = 0$, eqn (5) yields the usual NBC

$$L_{y'} = 0$$

Modified Brachistochrone



Transit time on C

$$T(y) = \gamma \int_0^t \sqrt{\frac{1+y'^2}{y}}\,dx$$

Here $L(x,y,z) = \sqrt{\frac{1+z^2}{y}}$. For <u>this</u> L

$$L f_y = L_{y'}(f_t + f_y y')$$

simplified becomes

(1) $$\boxed{f_y = y' f_t}$$

Using multivariate calculus we see the transversality condition (1) implies extrema intersect $f=0$ orthogonally!!

Note $\frac{d}{dt} f(t,y) = f_t + y' f_y = 0$ on $f=0 \Rightarrow y' = -f_t/f_y$. is tangent. Thus orthogonal slope

$$m = + \frac{f_y}{f_t}$$

which is (1) above.