Calculus of Variations - theory

Let V be a normed vector space.

$$J : \mathcal{A} \longrightarrow \mathbb{R}$$

where $\mathcal{A}$ is the admissible set.

Defn $\bar{y}$ is a global minima for J on $\mathcal{A}$ iff

$$J(\bar{y}) \leq J(y) \qquad \forall y \in \mathcal{A}$$

Defn $\bar{y}$ is a local minima for J if $\exists\, \delta > 0$ such that

$$\| y - \bar{y} \| < \delta \quad \Rightarrow \quad J(\bar{y}) \leq J(y)$$

Defn Suppose $\bar{y} \in \mathcal{A}$. We say δy is an admissible variation if

$$\bar{y} + \delta y \in \mathcal{A}$$

Some remarks on these definitions

1) $\mathcal{A}$ need not be a vector space

2) the local minima depends on the norm used

3) Often the space $\mathcal{A}^*$ of all admissible variations does not depend on the choice of $\bar{y}$

Gateaux Variation

There are different ways to define a derivative of $J(y)$

A Fréchet derivative of $J(y)$ is a linear operator $L(y)$ such that

$$(1) \quad J(y+\eta) = J(y) + L(y)\eta + R(\eta)$$

$$(2) \quad \lim_{\|\eta\| \to 0} \frac{\|R(\eta)\|}{\|\eta\|} = 0$$

providing the expressions are defined for all η sufficiently small.

Much, much more common is

Definition (First Gateaux variation)

Let $J: A \to \mathbb{R}$, $\bar{y} \in A$ and

$$\bar{y} + \varepsilon h \in A$$

for all $\varepsilon \in \mathbb{R}$ sufficiently small.
When it exists the first variation $\delta J(\bar{y}, h)$ of $J(y)$ at $\bar{y}$ is

$$\delta J(\bar{y}, h) \equiv \frac{d}{d\varepsilon} J(\bar{y} + \varepsilon h)\Big|_{\varepsilon = 0} = 0$$

Necessary Conditions for local minima

Let $J : A \to \mathbb{R}$, $\bar{y} \in A$ and assume $\bar{y} + \varepsilon h(x) \in A$ for all ε sufficiently small and $h(x)$ fixed.

$$F(\varepsilon) \equiv J(\bar{y} + \varepsilon h)$$

we assume to be twice continuously differentiable. Then

$$F(\varepsilon) = F(0) + F'(0)\varepsilon + \frac{F''(\zeta)}{2!}\varepsilon^2$$

for some $\zeta \in N_r(0)$. Alternately

$$(1) \quad J(\bar{y} + \varepsilon h) = J(\bar{y}) + \delta J(\bar{y}, h)\varepsilon + O(\varepsilon^2)$$

If $\bar{y}$ is a local min of J then

$$\Delta J = J(\bar{y} + \varepsilon h) - J(\bar{y}) \geq 0$$

for all ε sufficiently small. Given (1)

<u>Theorem</u>: $\bar{y}$ is a local minima of $J : A \to \mathbb{R}$ only if

$$\delta J(\bar{y}, h) = 0$$

for all admissible variations $\delta y = \varepsilon h(x)$.

EXAMPLE $\quad J(y) \equiv y(0)^4 \qquad \mathcal{A} = C[-1,1]$

Admissible variation are those $h(x)$ such that $y+h \in \mathcal{A}$ whenever y is in $\mathcal{A}$. Here

$$\mathcal{A}^* = C[-1,1]$$

Clearly if $y \in \mathcal{A}$ and $h \in \mathcal{A}^*$ then $y+h \in \mathcal{A}$.

To compute the First Variation

$$F(\varepsilon) = J(y+\varepsilon h)$$

$$F(\varepsilon) = (y(0)+\varepsilon h(0))^4$$

from which

$$F'(\varepsilon) = 4(y(0)+\varepsilon h(0))^3 h(0)$$

Evaluating at $\varepsilon = 0$

$$\delta J(y,h) = 4y(0)^3 h(0)$$

Recall y is a min only if $\delta J(y,h) = 0 \quad \forall h \in \mathcal{A}^*$
or

$$y(0) = 0$$

Should have been clear from start that

$$\min_{y \in \mathcal{A}} J(y) = 0$$

attained for any $y \in \mathcal{A}$ with $y(0) = 0$.

EXAMPLE Sturm Liouville Problems

$$J(y) \equiv \int_0^1 p(x)y'^2 + q(x)y^2\,dx$$

$$\mathcal{A} = \{y \in C^2[0,1] : y(0) = A,\ y(1) = B\}$$

First we note admissible variations $h(x)$ must vanish at the endpoints

$$\mathcal{A}^* = \{h \in C^2[0,1] : h(0) = h(1) = 0\}$$

Next we compute the First Variation

$$F(\varepsilon) = J(y + \varepsilon h)$$

$$F(\varepsilon) = \int_0^1 p(y' + \varepsilon h')^2 + q(y + \varepsilon h)^2\,dx$$

Taking the derivative in ε

$$F'(\varepsilon) = \int_0^1 2p(y' + \varepsilon h')h' + 2q(y + \varepsilon h)h\,dx$$

Hence $\delta J(y,h) = F'(0)$ is

$$\delta J(y,h) = \int_0^1 (2py'h' + 2qyh)\,dx$$

In this form it is difficult to conclude anything about $y(x)$. We next IBP first term

Integrating by Parts

$$\delta J(y,h) = \underbrace{\cancel{2py'h}\Big|_0^1}_{h\in A^*} + 2\int_0^1 [-(py')' + qy]h\,dx$$

Boundary conditions $h(0)=h(1)=0$ assure first term vanishes:

$$\delta J(y,h) = 2\int_0^1 \underbrace{[-(py')' + qy]}\, h(x)\,dx$$

For y to minimize J must have $\delta J = 0$ for all $h \in A^*$. By a theorem we state later, the indicated term must vanish so that minimizers are solutions to

$$(1) \qquad -(py')' + qy = 0 \qquad y(0)=A,\ y(1)=B$$

This is a SLP with nonhomogeneous boundary conditions!

Note (1) has a unique solution if $p(x) > 0$. Thus, if $J(y)$ has a minima, it is unique. Certainly if $p(x) > 0$ and $q(x) \geq 0$ this is the case!

Theorem Let $g(x) \in C[a,b]$ and suppose

$$(1) \qquad \int_a^b g(x)\, v(x)\, dx = 0 \,, \qquad \forall v \in A^*$$

where A^* is any one of the following spaces

$$A^* = C[a,b]$$

$$A^* = C^n[a,b]$$

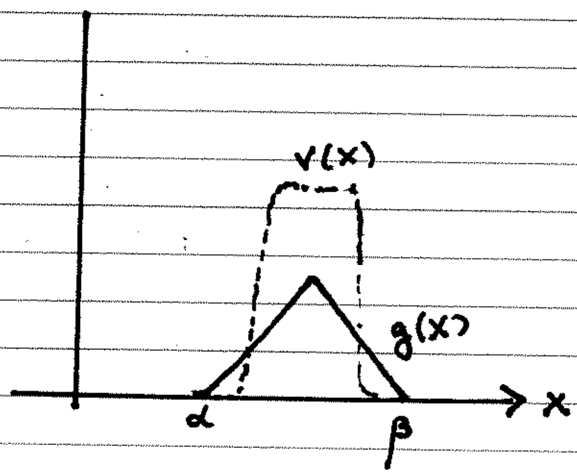
$$A^* = \{ y \in C^n[a,b] : y(a) = y(b) = 0 \}$$

Then $g(x) \equiv 0$

Proof outline (by contradiction)

Suppose (1) is true but $g(x) \not\equiv 0$. Since $g(x)$ is continuous there must be some interval $I = [\alpha, \beta] \subset [a,b]$ on which $g(x)$ has one sign. wlog $g(x) > 0$.

Pick any $v \in A^*$ positive on I but zero elsewhere.



Clearly

$$\langle g, v \rangle > 0$$

which is a contradiction.

Euler Lagrange Equations

Many variational problems have functionals

$$J(y) \equiv \int_a^b L(x, y, y')\, dx$$

$$\mathcal{A} = \{ y \in C^2[a,b] : y(a) = A, y(b) = B \}$$

Here $L(x,y,y')$ is the Lagrangian

Suppose $\bar{y} \in \mathcal{A}$ is the minimizer (maximizer or extrema) then

$$\mathcal{A}^* = \{ h \in C^2[a,b] : h(a) = h(b) = 0 \}$$

is the set of admissible variations and $\bar{y} + h \in \mathcal{A}$ for all $h \in \mathcal{A}^*$.

Define $y = \bar{y} + \varepsilon h$ and

$$F(\varepsilon) = J(\bar{y} + \varepsilon h)$$

Differentiating $F(\varepsilon)$ in ε we have

$$F'(\varepsilon) = \int_a^b (L_y h + L_{y'} h')\, dx$$

Integrate by parts

$$F'(\varepsilon) = L_{y'}(x,y,y')h\Big|_a^b + \int_a^b \left(L_y - \frac{d}{dx} L_{y'}\right) h\, dx$$

Evaluate at $\varepsilon = 0$ to get first variation

$$\delta J(\bar{y}, h) = L_{y'}(x, \bar{y}, \bar{y}')h\Big|_a^b + \int_a^b \left(L_y - \frac{d}{dx} L_{y'}\right) h\, dx$$

$h \in A^*$ makes h vanish at $x = a, b$

$$\delta J(\bar{y}, h) = \int_a^b \underbrace{\left(L_y - \frac{d}{dx} L_{y'}\right)}_{} h\, dx$$

must vanish if $\delta J(\bar{y}, h) = 0$ for all $h(x)$ variations

Conclude necessary conditions for $\bar{y}(x)$

(1) $\quad L_y = \frac{d}{dx} L_{y'} \quad$ Euler Lagrange Eqn

(2) $\quad \bar{y}(a) = A, \; \bar{y}(b) = B \quad$ B.C. define $\bar{y} \in A$

Collectively (1)-(2) is a (nonlinear) 2 point boundary value problem.

Special Cases of EL-equations

(1) $$\boxed{L_y = \frac{d}{dx} L_{y'}}$$

Simplifies depending on $L(x, y, y')$

CONDITION	EL-Eqn	
$L_y \equiv 0$	$L_{y'}(x, y') = c$	ODE
$L_{y'} \equiv 0$	$L_y(x, y) = 0$	ALGEBRAIC
$L_x \equiv 0$	$L - y' L_{y'} = c$	ODE

The last term is called a <u>first integral</u> since its derivative is zero

$$\frac{d}{dx}(L - y' L_{y'}) = \cdots = y' \underbrace{\left(L_y - \frac{d}{dx} L_{y'}\right)}_{EL.} = 0$$

Most often the case to use first integrals for $L_y = 0$ and $L_x = 0$ cases.

Otherwise (1) expanded out is a <u>2nd order nonlinear BVP</u> ... very hard.

EXAMPLE $\quad L = \sqrt{x^2 + y'^2}$ $\qquad (L_y = 0)$

$$A = \{ y \in C^2[0,1] : y(0) = 0,\ y(1) = 1 \}$$

Since $L_y = 0$, $L_{y'}$ is a first integral

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{x^2 + y'^2}} \right) = 0 = L_y$$

Term in $(///)$ is constant. Solve for y' then

$$y' = \alpha x \qquad \alpha \in \mathbb{R}$$

$$y = \tfrac{1}{2}\alpha x^2 + \beta \qquad \beta \in \mathbb{R}$$

Apply B.C. $y(0) = 0$ and $y(1) = 1$ to find α, β:

$$y(x) = x^2$$

EXAMPLE $\quad L = x^2 y - y^2$ with $A = C[0,1]$. $\quad (L_{y'} = 0)$

Since $L_{y'} = 0$ we have

$$L_y = x^2 - 2y = 0$$

and conclude

$$y(x) = \tfrac{1}{2} x^2$$

Note: if A had B.C. defining it they have been incompatible with solution. Then you conclude there is no extrema.

EXAMPLE $\quad L = y^2 y'^2$ $\qquad (L_x = 0)$

$$A = \{ y \in C[0,1] : y(0) = 1, y(1) = 2 \}$$

Here $L_x = 0$ so there is a different first integral:

$$L - y' L_{y'} = -y^2 y'^2 = -\alpha^2 \qquad \alpha \in \mathbb{R}$$

Hence

$$yy' = \alpha$$

$$\frac{d}{dx}\left(\tfrac{1}{2}y^2\right) = \alpha$$

$$\tfrac{1}{2}y^2 = \alpha x + \beta \qquad \beta \in \mathbb{R}$$

$$y^2 = 2\alpha x + 2\beta$$

(1) $$y = \sqrt{2\alpha x + 2\beta}$$

For $y(x)$ in (1) to satisfy B.C., $\beta = \frac{1}{2}$, $\alpha = \frac{3}{2}$

(2) $$y(x) = \sqrt{3x + 1}$$

Candidate for a min, not max since $J(y) \to \infty$ for large $y(x)$.

Not using first integral looks like

$$L_y = \frac{d}{dx}(L_{y'})$$

$$2yy'^2 = 4yy'^2 + 2y^2y''$$

arguably harder to solve. No one way but try first integrals first.

Natural Boundary Conditions

Recall that y extremizes (max, min, neither) the functional $J(y)$ only if

$$\delta J(y,h) = 0 \qquad \forall h \in \mathcal{A}^*$$

and this statement is independent of how the admissible set $\mathcal{A}$ is defined. Suppose for example

$$J(y) = \int_a^b L(x,y,y')\,dx$$

$$\mathcal{A} = \{ y \in C^2[a,b] : \; y(a) = A \}$$

Here $\mathcal{A}$ has only one B.C. defining it and

$$\mathcal{A}^* = \{ h \in C^2[a,b] : \; h(a) = 0 \}$$

To find $\delta J(y,h)$ we differentiate

$$F(\varepsilon) = J(y + \varepsilon h)$$

evaluate at $\varepsilon = 0$ and integrate by parts:

$$\delta J = \underbrace{L_{y'}(x,y,y')h \Big|_a^b}_{\substack{h(a)=0 \text{ but} \\ h(b) \text{ can have} \\ \text{any value!}}} + \int_a^b \underbrace{\left(L_y - \frac{d}{dx} L_{y'}\right)}_{\substack{\text{must vanish} \\ \text{as well.}}} h\,dx$$

Explicitly, $h \in \mathcal{A}^* \Rightarrow h(a) = 0$ so that

$$\delta J = \underbrace{L_{y'}(b, y(b), y'(b))\, h(b)}_{T_1} + \underbrace{\int_a^b \left(L_y - \frac{d}{dx} L_{y'}\right) h \, dx}_{T_2}$$

This must vanish <u>for all</u> $h \in \mathcal{A}^*$. Consider $\mathcal{A}^{**}$ be the $h \in \mathcal{A}^*$ s.t. $h(b) = 0$ hence $T_1 = 0$ above. The second term T_2 must vanish $\forall h \in \mathcal{A}^{**}$ which is sufficient to conclude

$$(1) \qquad L_y = \frac{d}{dx} L_{y'}$$

Knowing now that (1) is necessary,

$$\delta J = L_{y'}(b, y(b), y'(b))\, h(b)$$

This must vanish $\forall h \in \mathcal{A}^*$ hence.

$$(2) \qquad L_{y'}(b, y(b), y'(b)) = 0 \qquad \text{N.B.C.}$$

which is a (possibly nonlinear) B.C. Collectively the extrema must solve

$$\boxed{\begin{array}{lll} L_y = \dfrac{d}{dx} L_{y'} & & \text{EL-eqns} \\[2ex] y(a) = A & & \text{Given B.C.} \\[2ex] L_{y'}\Big|_{x=b} = 0 & & \text{Natural B.C.} \end{array}}$$

Ex what B.V.P must the extrema of $J(y)$ satisfy if

$$J(y) = \int_0^1 L(x, y, y')\,dx$$

$$A = \{ y \in C^2[0,1] : y(1) = 7 \}$$

and $L = xy^2 + (y')^2$ is the Lagrangian.

$$L_y = 2xy$$

$$L_{y'} = 2y'$$

Here we apply previous theory. $y(1) = 7$ is a given B.C. The Natural Boundary Condition (NBC) is

$$L_{y'}\Big|_{x=2} = 2y'(2) = 0$$

Consequently the BVP the extrema must solve is

(1)	$xy = y''$	EL-eqn
(2)	$y(1) = 7$	Given B.C.
(3)	$y'(2) = 0$	N.B.C.

EX What B.V.P must extrema of $J(y)$ satisfy if

$$J(y) = y(2)^3 + \int_1^2 L(x,y,y')\,dx$$

$$\mathcal{A} = \{ y \in C^2[1,2] : y(1) = 3 \}$$

where $L = xy + (y')^3$. Given $\mathcal{A}$ we see

$$\mathcal{A}^* = \{ h \in C^2[1,2] : h(1) = 0 \}$$

For given Lagrangian, $J(y)$ first variation:

(1) $$\delta J = 3y(2)^2 h(2) + L_{y'} h \Big|_1^2 + \int_1^2 \left(L_y - \frac{d}{dx} L_{y'}\right) h\,dx$$

Since $L_{y'} = 3y'^2$ and $h(1) = 0$, eqn (1) becomes

$$\delta J = \underbrace{3\left(y(2)^2 + y'(2)^2\right)}_{=0 \quad \text{NBC}} h(2) + \int_1^2 \underbrace{\left(L_y - \frac{d}{dx} L_{y'}\right)}_{=0 \quad \text{EL-eqn}} h\,dx$$

Condition $\delta J = 0 \ \forall h \in \mathcal{A}^*$ implies

(1) $\quad x = \frac{d}{dx}\left(3y'^2\right) \quad$ EL-eqn

(2) $\quad y(1) = 3 \quad$ Given B.C.

(3) $\quad y(2)^2 + y'(2)^2 = 0 \quad$ N.B.C.

Remark: Solving (1) is possible and yields $y = y(x; c_1, c_2)$ where c_K are constants. Eqns (2)-(3) then are two eqns for (c_1, c_2). If there is no soln there is no extrema. Here, extremely hard.

Euler Lagrange Domain issues

whether extrema exist or are unique depends very much on $\mathcal{A}$. Even if extrema exist they may or may not be (local) minima.

EXAMPLE

$$J(y) \equiv \int_0^1 [y'^2 - 1]^2 dx$$

$$\mathcal{A} \equiv \{ y \in C^1[0,1] : y(0) = y(1) = 0 \}$$

Since $L_y = 0$ the Euler-Lagrange eqns are

$$L_{y'} = 4y'(y'^2 - 1) = c \qquad \forall x \in [0,1]$$

where $c \in \mathbb{R}$ constand. Has form $P(z) = c$ where $P(z) = 4z(z^2-1)$. Since P is cubic $\exists z_1 \in \mathbb{R}$ s.t $P(z_1) = c$ hence

$$y'(x) = z_1$$

$$y(x) = z_1 x + z_2$$

Since $y(0) = y(1) = 0$ we conclude $z_1 = z_2 = 0$

$$\bar{y}(x) \equiv 0$$

This is the extrema over $\mathcal{A}$ and

$$J(\bar{y}) = 1$$

But it should be clear one can make $J(y)$ smaller for other functions

For instance, note

$$J(y) = \int_0^1 (y'^2 - 1)^2 dx \geq 0$$

and $J(y) = 0$ if

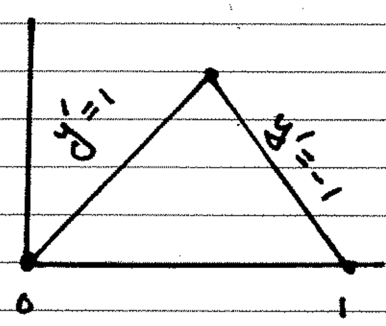
$$(y')^2 = 1 \qquad \forall x \in [0,1]$$

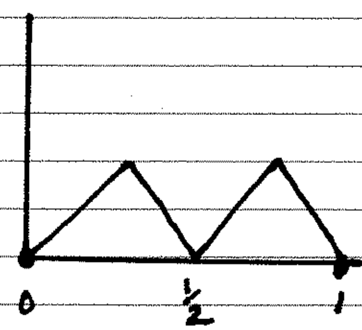
There does not exist a $y \in \mathcal{A}$ such that this is true. However suppose we extremize over

$$\mathcal{A} = \{ y \in PC^1[0,1] : y(0) = y(1) = 0 \}$$

Then the min <u>is</u> attained but the minimizer is (badly) non unique.

Here's two graphical examples.





EXAMPLE <u>Extrema versus minima</u>

$$J(y) = \int_1^2 x(y')^2\,dx$$

$$\mathcal{A} = \{ y \in C^1[1,2] : \ y(1) = 0, \ y'(2) = 1 \}$$

Euler-Lagrange eqns are, for $c \in \mathbb{R}$,

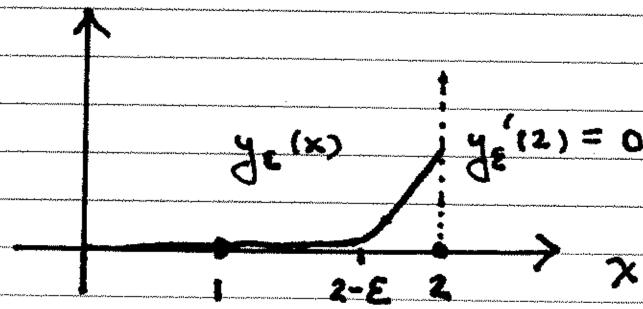
$$2xy' = c \qquad\qquad y(1) = 0 \quad y'(2) = 1$$

whose soln is

$$\bar{y}(x) = 2\ln x$$

One can compute $J(\bar{y}) = \ln 2 > 0$.

Although $\bar{y}(x)$ is an extrema, it is not a minima. Consider $y_\varepsilon(x)$ graphed below



Note that $y_\varepsilon(x) \in \mathcal{A}$ for all $\varepsilon < 1$ and

$$J(y_\varepsilon) \to 0 \qquad\qquad \text{as } \varepsilon \to 0$$

so $J(y_\varepsilon) < J(\bar{y})$. The infimum above is not attained:

$$\nexists\, y^* \in \mathcal{A} \ \text{ s.t. } J(y^*) = 0.$$

EXAMPLE <u>Nonuniqueness of extrema</u>

$$J(y) = \int_0^1 (y')^2 \, dx$$

$$A = \{ y \in C^1[0,1] : y'(0) = y'(1) = 0 \}$$

Any extrema (min/max) must satisfy the EL-eqns

$$L_y = \frac{d}{dx} L_{y'}$$

$$0 = \frac{d}{dx}(2y')$$

Hence $y(x)$ is a linear function

$$y(x) = Ax + B$$

Both boundary conditions are satisfied if $A = 0$. Hence

$$y(x) = B \qquad B \in \mathbb{R}$$

are a family of extrema

When problems do not have unique solns. they are said to be <u>ill-posed</u>

Higher order derivatives

$$J(y) \equiv \int_a^b L(x, y, y', y'')\,dx$$

Let $y \in \mathcal{A}$, $h \in \mathcal{A}^*$. Then

$$F(\varepsilon) \equiv J(y + \varepsilon h)$$

y is an extrema of $J(y)$ only if

$$F'(0) = \delta J(y, h) = 0 \qquad \forall h \in \mathcal{A}^*$$

Can easily show

$$F'(\varepsilon) = \int_a^b (L_y h + L_{y'} h' + L_{y''} h'')\,dx$$

Integrate by parts (twice) and set $\varepsilon = 0$:

$$(1) \qquad \delta J = B(y, h) + \int_a^b \underbrace{\left(L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''}\right)}_{\text{EL-eqns}} h\,dx$$

where the boundary term is

$$(2) \qquad B = \left(L_{y'} - \frac{d}{dx} L_{y''}\right) h + L_{y''} h' \Bigg|_{x=a}^{x=b}$$

Independent of B.C. defining $\mathcal{A}$, must have

$$(3) \qquad \boxed{L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} = 0}$$

is the EL-eqn for the variation problem.

Depending on the given B.C. defining A, $B(y,h)$ must vanish $\forall h \in A^*$. This will detemine Natural B.C.

For example if

$$A = \{ y \in C^4[0,1] : \ y(0) = 7 \ , \ y(1) = 2 \}$$

then

$$A^* = \{ h \in C^4[0,1] : \ h(0) = h(1) = 0 \}$$

Given $B(y,h)$ in eqn 2 we have

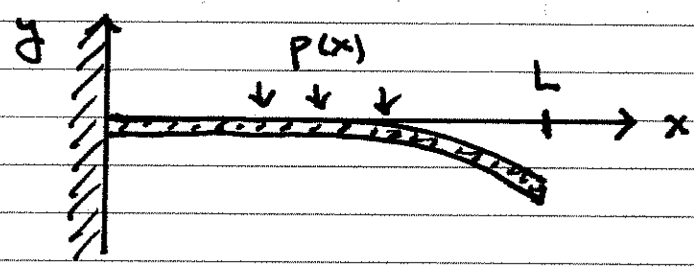
$$B = L_{y''} h' \Big|_{x=a}^{x=b}$$

Since $h'(a)$, $h'(b)$ are arbitrary the natural B. Cond are

$$L_{y''}(0, y(0), y'(0), y''(0)) = 0$$

$$L_{y''}(1, y(1), y'(1), y''(1)) = 0$$

EXAMPLE Beam Deflection (Elastic)



Shape minimizes potential energy

$$V(y) = \int_0^L \left(\tfrac{1}{2}\mu (y'')^2 - p(x)\, y(x) \right) dx$$

where

$$\mu = \text{flexural rigidity (constant)}$$

$$p(x) = \text{load } (N/m)$$

The beam is pinned at $x=0$ so $y(0)=0$. Also, there is no moment at $x=0$ so $y'(0)$

$$\mathcal{A} = \{ y \in C^4(0,L) : \; y(0) = y'(0) = 0 \}$$

and

$$\mathcal{A}^* = \mathcal{A}$$

In this problem the Lagrangian is

$$L(x,y,y'') = \tfrac{1}{2}\mu (y'')^2 - p(x)\, y$$

Note $L_{y'} \equiv 0$.

Compute partials:

$$L_y = -p \qquad L_{y'} = 0 \qquad L_{y''} = \mu y''$$

Given general theory for B. Cond $B(y, h)$.

$$B = \underbrace{-\mu y''' h} + \underbrace{\mu y'' h'}\Big|_{x=1}$$

Indicated terms vanish independently.

$y^{(4)}(x) = p(x)$	EL-Eqn	
$y(0) = 0$	BC	pinned
$y'(0) = 0$	BC	no moment
$y''(1) = 0$	NBC	no bending moment
$y'''(1) = 0$	NBC	no shear force

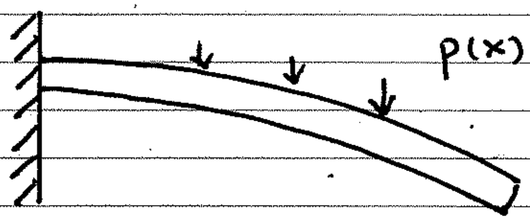
Constant Load Case (p constant) and $L = 1$

$$y = \frac{p}{24} x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4$$

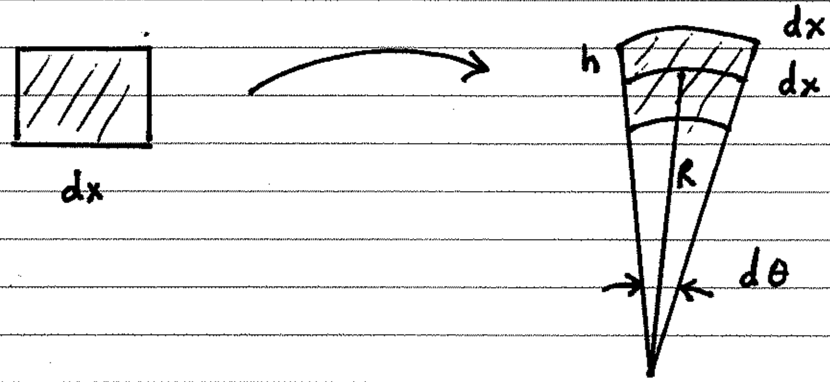
Boundary conditions determine c_k and

$$y(x) = \frac{1}{24} p x^2 (x^2 - 4x + 6)$$

Elastic Beam : Addendum on elatic energy



Energy of an element under deformation



Here R is the radius of curvature. Then

$$dx = R\,d\theta \qquad dx' = (R+h)\,d\theta$$

Yields

$$dx' = \left(1 + \frac{h}{R}\right) dx$$

and the change in displacement length

$$dx' - dx = \frac{h}{R}\,dx = h\,\kappa(x)\,dx$$

where the local curvature $\kappa = R^{-1}$ is

$$\kappa(x) = \frac{y''}{\sqrt{1+y'^2}}$$

For linear elasticity (Hookes Law) the energy of deformation is proportional to the displacement squared. Hence such energy should be proportional to

$$\kappa^2 = \frac{(y'')^2}{1 + y'^2}$$

For small displacements $\kappa^2 = (y'')^2 + o(1)$ and we find

$$E = \int_0^L \underbrace{\tfrac{1}{2}\mu (y'')^2}_{\text{elastic}} - \underbrace{p(x)\, y}_{\text{load}} \, dx$$