Distribution Theory

A functional $\hat{t}$ is a mapping $\hat{t}: D \to \mathbb{R}$ where D is a test function space. Typically

$$D = \{ \phi \in C^\infty(\Omega) : \text{supp}\,\phi \text{ compact} \}$$

Linear functionals on D are continuous if

$$\hat{t}(\phi_n) \to 0$$

for all zero sequences $\{\phi_n\}$ satisfying

i) $\bigcup_{n \geq 1} \text{supp}\,\phi_n$ bounded

ii) $\lim_{n \to \infty} \left(\max \left| \frac{\partial^k \phi_n}{\partial x^k} \right| \right) = 0$ for $k = 0, 1, 2 \ldots$

EXAMPLE A test function in $D = C^\infty(\mathbb{R})$

$$\phi(x, a) = \begin{cases} \exp\left(\frac{a^2}{x^2 - a^2}\right) & |x| < a \\ 0 & |x| \geq a \end{cases}$$

EXAMPLE A zero sequence

$$\phi_n(x) \equiv \frac{1}{n} \phi(x, a)$$

Defn A distribution $\hat{t}$ is a continuous linear functional

(1) $\hat{t} : D \to \mathbb{R}$

(2) $\langle \hat{t}, \alpha_1\phi_1 + \alpha_2\phi_2 \rangle = \alpha_1 \langle \hat{t}, \phi_1 \rangle + \alpha_2 \langle \hat{t}, \phi_2 \rangle$

(3) $\langle \hat{t}, \phi_n \rangle \to 0$ $\forall$ zero sequences $\{\phi_n\}$

Regular versus singular distributions

$L^2_{loc} = \{ u : u \in L^2(I)$ for all bounded open I

For instance $u(x) = x^2$, $u \in L^2_{loc}$ but $u \notin L^2(\mathbb{R})$

Defn: $\hat{t}$ is a regular dist^n if $\exists\, t \in L^2_{loc}(\Omega)$ s.t.

$$\langle \hat{t}, \phi \rangle = \int_\Omega t(x)\phi(x)\,dx \qquad \forall \phi \in D$$

$\hat{t}$ is a singular dist^n if it is not regular

Ex $f(x) = x^2$ generates a dist^n (regular) $\hat{f}$ via

$$\langle \hat{f}, \phi \rangle \equiv \int_{\mathbb{R}} f(x)\phi(x)\,dx$$

Ex Dirac Delta dist^n is singular. Defined by

$$\langle \delta, \phi \rangle \equiv \phi(0) \qquad \forall \phi \in D$$

Clearly $\langle \delta, \phi_n \rangle = \phi_n(0) \to$ for all zero sequences.
Doesn't exist a function $\delta(x)$ s.t.

$$\int_{\mathbb{R}} \delta(x)\phi(x)\,dx = \phi(0) \qquad \forall \phi$$

$\exists\, I$ s.t. integral $\delta(x)$ is one sign. Pick $\phi > 0$ on I having $\phi(0) = 0$ for contradiction. Singular!

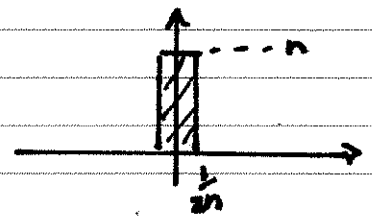
Delta Sequences

Any sequence $\{\delta_n\} \subset L^2_{loc}(\Omega)$ such that

$$\lim_{n\to\infty} \int_\Omega \delta_n(x)\,\phi(x)\,dx = \phi(0)$$

EXAMPLES

a) $$\delta_n(x) = \begin{cases} n & |x| \le \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$



b) $$\delta_n(x) = \frac{n}{\pi(1+n^2x^2)}$$

c) $$\delta_n(x) = \left(\frac{n}{\pi}\right)^{1/2} \exp(-nx^2)$$

d) $$\delta_n(x) = \frac{\sin(nx)}{\pi x}$$

Ex Can construct higher dimensional δ fns.

$$\delta(\vec{x}) = \delta(x)\,\delta(y)$$

where $\langle \delta, \phi \rangle = \phi(0,0)$ for $\phi \in D$. Using form a) above

$$\delta_n(\vec{x}) = \delta_n(x)\,\delta_n(y)$$

has property

$$\lim_{n\to\infty} \iint_{\mathbb{R}^2} \delta_n(\vec{x})\,\phi(\vec{x})\,d\vec{x} = \phi(0,0)$$

EX **Heaviside Dist^n $\hat{H}$** (regular)

$$\langle \hat{H}, \phi \rangle \equiv \int_{\mathbb{R}} H(x)\phi(x)\,dx = \int_0^\infty \phi(x)\,dx$$

EX **Principal Value integrals**

Note $f(x) = \frac{1}{x}$ does not define reg. dist^n since

$$\int_{\mathbb{R}} \frac{\phi(x)}{x}\,dx$$

is nonconvergent $\forall \phi$. Define Principal Value integral as a (singular) dist^n

$$\langle Pf(\tfrac{1}{x}), \phi \rangle \equiv \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\phi(x)}{x}\,dx$$

which always exists for $\phi \in D$ as can be seen from, for some $\psi \in D$,

$$\langle Pf(\tfrac{1}{x}), \phi \rangle = \lim_{\varepsilon \to 0} \left(\int_{|x| \geq \varepsilon} \frac{\phi(0) + x\psi(x)}{x}\,dx \right)$$

$$= \cancel{\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\phi(0)}{x}}^{\,0} + \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \psi(x)\,dx$$

The second is finite. And, were $\phi = \frac{x^2}{x^2+1}$ we find $\psi = \frac{x}{x^2+1}$ so that, for instance,

$$\lim_{\varepsilon \to 0} \int_{1 \geq x \geq 0} \psi(x) = \tfrac{1}{2}\ln 2 = \langle Pf(\tfrac{1}{x}), \phi \rangle$$

EX Expected value distribution of rand. var $\mathbb{X}$ with density function $p(x)$

$$P(a < \mathbb{X} < b) = \int_a^b p(x)\,dx$$

Then expected value of $\mathbb{X}$ is

$$\langle E, \phi \rangle \equiv \int_{\mathbb{R}} x\,\phi(x)\,dx$$

EX The function $f(x) = H(x)\log x$ defines a distribution

$$\langle f, \phi \rangle = \int_0^\infty \phi(x)\log(x)\,dx$$

since $\log(x)$ has an integrable singularity

EX Delta function $\delta(\vec{x}-\vec{x}_0)$ on $\mathbb{R}^2$ in polar where $\vec{x}, \vec{x}_0$ are cartesian. By defn

$$\langle \delta(\vec{x}-\vec{x}_0), \phi \rangle = \phi(\vec{x}_0) \qquad \forall \phi \in D$$

Here $\phi \in C^\infty(\mathbb{R}^2)$ have compact support.

$$\delta(\vec{x}-\vec{x}_0) = \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)}{r}$$

in the sense

$$\langle \delta(\vec{x}-\vec{x}_0), \phi \rangle = \iint_{\mathbb{R}^2} \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)}{r}\,\phi(r,\theta)\,\underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_0^{2\pi} \delta(\theta-\theta_0) \int_0^\infty \delta(r-r_0)\,\phi(r,\theta)\,dr\,d\theta$$

$$= \phi(r_0,\theta_0) = \phi(\vec{x}_0)$$

EX $\boxed{\text{Show } \Delta\left(\frac{1}{|x|}\right) = -4\pi\delta(x) \qquad x \in \mathbb{R}^3}$

Since Δ is formally <u>self adjoint</u>, for any $\phi \in D$

(1) $$\left\langle \Delta\left(\tfrac{1}{|x|}\right), \phi \right\rangle = \left\langle \tfrac{1}{|x|}, \Delta\phi \right\rangle = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\Delta\phi}{|x|}\,dx$$

Use Green's 2nd identity to expand RHS

(2) $$\int_{|x|>\varepsilon} \frac{\Delta\phi}{|x|}\,dx = \int_{|x|>\varepsilon} \phi\,\Delta\left(\tfrac{1}{|x|}\right)dx + \int_{|x|=\varepsilon}\left(\frac{1}{|x|}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)\right)ds$$

The outward normal for $|x|>\varepsilon \Rightarrow \frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$.
Since $\Delta\left(\frac{1}{|x|}\right) = 0$ for $|x| \neq 0$, eqn (2) becomes

(3) $$\int_{|x|>\varepsilon} \frac{\Delta\phi}{|x|}\,dx = -\int_{\underset{\uparrow\,\text{bnd}}{|x|=\varepsilon}} \Big(\frac{1}{r}\frac{\partial\phi}{\partial r} + \overset{*}{\frac{\phi}{r^2}}\Big)ds$$

Indicated term vanishes as $\varepsilon\to 0$. Let $\left|\frac{\partial\phi}{\partial r}\right| \le M$ then

$$\left|\int_{|x|=\varepsilon} \frac{1}{r}\frac{\partial\phi}{\partial r}\,ds\right| \le \frac{M}{\varepsilon}(4\pi\varepsilon^2) \to 0$$

Only term $*$ matters in (3) as $\varepsilon \to 0$

$$\int_{|x|=\varepsilon} \frac{\phi}{r^2}\,ds = \int \frac{\phi(0) + \overbrace{(\phi(x)-\phi(0))}^{\text{(trick)}}}{r^2}\,ds = 4\pi\phi(0) + \int_{|x|=\varepsilon} \overbrace{\cancel{\frac{\phi(x)-\phi(0)}{r^2}}}^{0 \text{ by conty of } \phi(0)}\,ds$$

Hence (1) becomes

$$\left\langle \Delta\left(\tfrac{1}{|x|}\right), \phi \right\rangle = -4\pi\phi(0) \qquad \forall \phi \in D$$

Distribution Properties

The following hold for all regular distn^s and are defined to be true for singular distributions.

Let $\phi \in D$, $\hat{t} : D \to \mathbb{R}$ and $\psi \in L^2_{loc}(\Omega)$

1) The product $\psi \hat{t}$ is the dist^n s.t.

$$\langle \psi \hat{t}, \phi \rangle = \langle \hat{t}, \psi \phi \rangle \qquad \forall \phi \in D$$

2) The distribution $\hat{t}(\alpha x)$, $\alpha \in \mathbb{R}$ is defined by

$$\langle \hat{t}(\alpha x), \phi \rangle = \frac{1}{|\alpha|} \langle t, \phi(\tfrac{x}{\alpha}) \rangle$$

3) The distribution $\hat{t}(x-\xi)$, $\xi \in \mathbb{R}$:

$$\langle \hat{t}(x-\xi), \phi \rangle = \langle \hat{t}, \phi(x+\xi) \rangle$$

4) The distribution $\hat{t}^{(n)}$ n-derivative :

$$\langle \hat{t}^{(n)}, \phi \rangle = (-1)^n \langle \hat{t}, \phi^{(n)} \rangle$$

EXAMPLE Consider 2) and 4) for regular dist^{n}s

$$\langle t(\alpha x), \phi \rangle = \int_{\mathbb{R}} t(\alpha x) \phi(x) dx = \int_{\mathbb{R}} t(z) \frac{\phi(z/\alpha)}{\alpha} dz \qquad (\alpha > 0)$$

$$\langle t'(x), \phi \rangle = \int_{\mathbb{R}} t'(x) \phi(x) dx = \cancel{t\phi\Big|_{-\infty}^{\infty}}^{\,0 \text{ compact support}} - \int_{\mathbb{R}} t(x) \phi'(x) dx$$

Hence $\langle t', \phi \rangle = -\langle t, \phi' \rangle$. Induction for 4)

EXAMPLE $H'(x) = \delta(x)$ for Heaviside Dist^n $H(x)$

$$\langle \underline{\underline{H}}', \phi \rangle \equiv - \langle H, \phi' \rangle$$

$$= - \int_0^\infty \phi'(x)\, dx$$

$$= + \phi(0)$$

$$= + \langle \underline{\underline{\delta}}, \phi \rangle \qquad H'(x) = + \delta(x)$$

EXAMPLE Show $\delta(3x+4) = \frac{1}{3}\delta(x + \frac{4}{3})$

$$\langle \delta(3x+4), \phi \rangle = \int_{\mathbb{R}} \delta(z)\, \phi\left(\frac{z-4}{3}\right) \cdot \frac{1}{3}\, dz$$

$$= \frac{1}{3}\phi(-\tfrac{4}{3})$$

$$= \langle \tfrac{1}{3}\delta(x + \tfrac{4}{3}), \phi \rangle$$

EXAMPLE Show $x\delta'(x) = -\delta(x)$

$$\langle \underline{\underline{x\delta'}}, \phi \rangle = \langle \delta', x\phi \rangle$$

$$= -\langle \delta, (x\phi)' \rangle$$

$$= -\langle \delta, \phi + x\phi' \rangle$$

$$= -\phi(0)$$

$$= -\langle \underline{\underline{\delta}}, \phi \rangle$$

EXAMPLE $\cos x\, \delta(x) = \delta(x)$ distributionally

Algebraic Problem: Distributional Soln

Let $\hat{t}(x)$ be a dist^n and $\delta(x)$ the delta fn. We say (for example) $\hat{t}(x)$ is a distributional solution of

$$(1) \qquad x\,\hat{t}(x+\xi) = g(x) + \delta(x)$$

iff

$$(2) \qquad \langle x\,\hat{t}(x+\xi), \phi \rangle = \langle g+\delta, \phi \rangle \qquad \forall \phi \in D$$

EXAMPLE $\quad \boxed{(x-\xi)\,\hat{t}(x-\xi) = g(x)}$

Homogeneous soln $\hat{t}_h$ with $g=0$ is $\hat{t}_h = \delta(x-\xi)$ since

$$\langle (x-\xi)\delta(x-\xi), \phi \rangle = \langle \delta(x-\xi), (x-\xi)\phi \rangle = 0 \qquad \forall \phi \in D$$

Claim a particular soln $\hat{t}_p$ is

$$\hat{t}_p = g(x)\,\mathrm{Pf}\left(\frac{1}{x-\xi}\right)$$

Verify

$$\langle (x-\xi)\hat{t}_p, \phi \rangle = \left\langle \mathrm{Pf}\left(\frac{1}{x-\xi}\right), (x-\xi)\,g(x)\,\phi(x) \right\rangle$$

$$= \lim_{\varepsilon \to 0} \int_{|x-\xi| \geq \varepsilon} g(x)\,\phi(x)$$

regular inner product (reg. dist^n)

$$= \langle g, \phi \rangle$$

Dist^n soln.

$$\hat{t}(x-\xi) = \delta(x-\xi) + g(x)\,\mathrm{Pf}\left(\frac{1}{x-\xi}\right)$$

Differential Equation: Distributional Solns

As an example $\hat{t}$ would be a sdistributional solution of

$$\hat{t}'' + \hat{t}' + \hat{t} = \delta(x)$$

only if

$$\langle \hat{t}'' + \hat{t}' + \hat{t}, \phi \rangle = \phi(0) \qquad \forall \phi \in D$$

EXAMPLE Show $\hat{t}(x) = H(x)$ is the dist^ soln of

$$(1) \qquad x\,\hat{t}'(x) = 0$$

Need to show $\langle x\hat{t}', \phi \rangle = 0 \quad \forall \phi \in D$.

$$\langle x\hat{t}', \phi \rangle = \langle \hat{t}', x\phi \rangle$$

$$= -\langle \hat{t}, (x\phi)' \rangle$$

$$= -\langle \hat{t}, x\phi' + \phi \rangle$$

At this stage we assume $\hat{t}(x) = H(x)$ Heaviside

$$= -\langle H, \phi \rangle - \int_0^\infty x\phi'(x)\,dx$$

$$= -\langle H, \phi \rangle - \cancel{x\phi}\Big|_0^\infty + \int_0^\infty \phi(x)\,dx$$

$$= -\langle H, \phi \rangle + \langle H, \phi \rangle$$

$$= 0$$