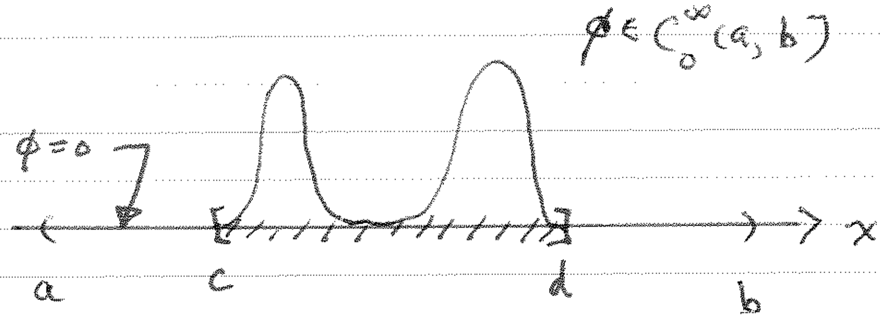
Test Functions $\quad u : \mathbb{R} \to \mathbb{R}$.

$$C^\infty(a,b) = \{ u(x) : \quad \forall n \geq 0 ,\ u^{(n)}(x) \text{ cont. on } (a,b) \}$$

Derivatives of all orders are continuous on (a,b). These are very smooth functions. Also, $(a,b) = \mathbb{R}$ is possible.

Defn The space of test functions $C_0^\infty(a,b)$ are $C^\infty(a,b)$ functions of compact support, i.e zero outside of some $[c,d] \subset (a,b)$.

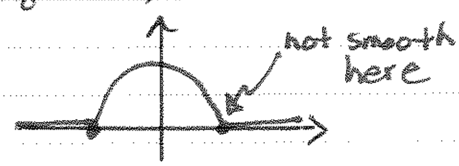
$$C_0^\infty(a,b) = \{ u \in C^\infty(a,b) : \exists [c,d] \subset (a,b) \ni x \notin [c,d] \Rightarrow u(x) = 0$$



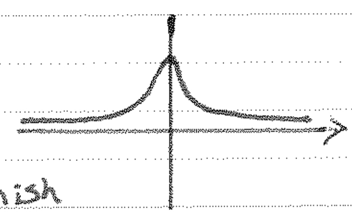
NOTE: $\phi, \phi^{(n)}$ must vanish at $x = a, b$

EXAMPLE Functions not in $C_0^\infty(\mathbb{R})$

$$u(x) = \begin{cases} 1 - x^2 & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$



$$u(x) = e^{-x^2}$$



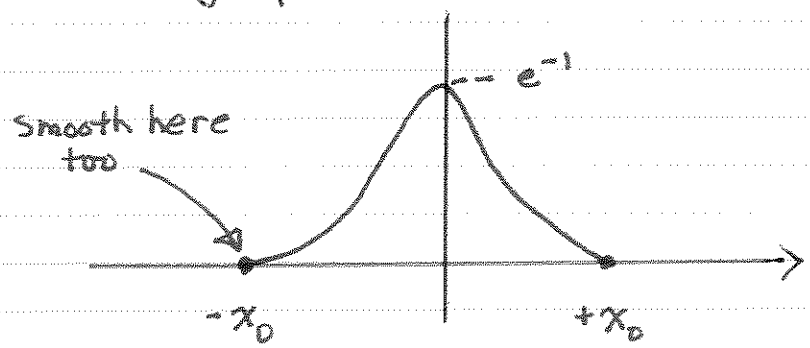
doesn't vanish outside any $[c,d]$

EXAMPLE Explicit formula of a test fn.

$$u(x) = \begin{cases} \exp\left(\dfrac{-x_0^2}{x_0^2 - x^2}\right) & |x| < x_0 \\ 0 & |x| \geq x_0 \end{cases}$$

has the graph

"Bump Function"



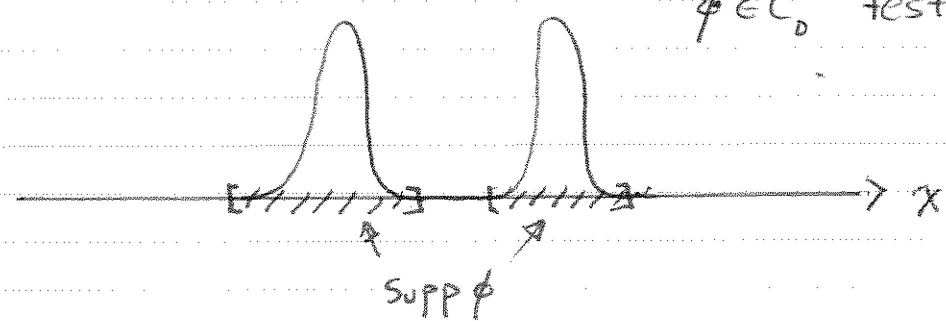
One can prove $u^{(n)}(x_0) = 0 \ \forall n$ in particular!

Support of (any) function

The support supp u of $u(x)$ is the closure of the set of x where $u(x) \neq 0$

$$\text{supp } u \equiv \overline{\{x : u(x) \neq 0\}}$$

$\phi \in C_0^\infty$ test fn



Weak Derivative Motivation/Defn

Before we begin note that if $\phi \in C_0^\infty(a,b)$ is a test function

$$\phi(a) = \phi(b) = 0$$

Now consider any $u(x) \in C^1(a,b)$ with derivative $f(x)$

$$(1) \qquad u'(x) = f(x)$$

Multiply by $\phi \in C_0^\infty(a,b)$ and integrate

$$\int_a^b u'(x)\phi(x)\,dx = \int_a^b f(x)\phi(x)\,dx$$

Integrate the left side by parts ($\phi(a) = \phi(b) = 0$)

$$(2) \qquad -\int_a^b u(x)\phi'(x)\,dx = \int_a^b f(x)\phi(x)\,dx$$

We use this to motivate our defn of a weak derivative

Defn We say $f(x)$ is a weak derivative of $u(x)$ on (a,b) if

$$\langle f, \phi \rangle = -\langle u, \phi' \rangle \qquad \forall \phi \in C_0^\infty(a,b)$$

Remark If $u \in C^1$ then $u'(x)$ is its weak derivative. Some functions that are not continuously differentiable still have weak derivatives though!

There are more functions having weak derivatives.

EXAMPLE $u(x) = |x| \qquad f(x) = H(x) - H(-x)$

The regular derivative $u'(x)$ exists $\forall x \neq 0$. Formally $f(x)$ is the weak derivative of $u(x)$.

$$\langle f, \phi \rangle = -\int_{-1}^{0} \phi \, dx + \int_{0}^{1} \phi \, dx$$

$$= \cancel{-x\phi\Big|_{-1}^{0}} + \int_{-1}^{0} x\phi' dx \cancel{+ x\phi\Big|_{0}^{1}} \int_{0}^{1} x\phi' dx$$

$$= \int_{-1}^{0} x\phi' dx - \int_{0}^{1} x\phi' dx$$

$$= -\int_{-1}^{0} |x|\phi' dx - \int_{0}^{1} |x|\phi' dx$$

$$= -\int_{-1}^{1} |x|\phi'(x) dx$$

$$= -\langle |x|, \phi' \rangle \qquad \forall \phi \in C_0^\infty(-1,1)$$

Distributions

Defn A distribution T on $D = C_0^\infty(a,b)$ is a functional satisfying

(1) $T : D \to \mathbb{R}$

(2) $T(a_1\phi_1 + a_2\phi_2) = a_1 T(\phi_1) + a_2 T(\phi_2) \quad \forall a_k \in \mathbb{R},\ \phi_k \in D$

(3) $\lim_{n\to\infty} T(\phi_n) = 0$ for all $\phi_n \to 0$

These properties are, in words,

(1) T is a functional

(2) T is a linear functional

(3) T is continuous

Defn A distribution is said to be regular if there exists a (locally integrable) function $f(x)$ such that

$$T(\phi) = \int_a^b f(x)\phi(x)\,dx \qquad \forall \phi \in D$$

If T is not regular it is said to be singular

Notations

Every locally integrable function $f(x)$ defines a distribution

$$(1) \quad T_f(\phi) \equiv \langle f, \phi \rangle = \int_a^b f(x)\phi(x)\,dx$$

where $\langle f, \phi \rangle$ is the usual inner product notation on $L^2(a,b)$.

whether a distribution is regular or not it is common to adopt the notation

$$T(\phi) = \langle T, \phi \rangle$$

as the value of T at ϕ.

The reason for this is that the definition of distributional derivatives (in particular) will be defined from analogies with inner product properties?

EXAMPLE Dirac delta (function) distribution

The Dirac delta distribution δ is defined by

(1) $$\langle \delta, \phi \rangle \equiv \phi(0) \qquad \forall \phi \in D$$

Thus δ merely evaluates ϕ at $x=0$.

The inner product notation suggests we may write

(2) $$\langle \delta, \phi \rangle = \phi(0) = \int_a^b \delta(x)\phi(x)\,dx$$

for some real function $\delta(x)$. However, it can be shown there is no function $\delta : \mathbb{R} \to \mathbb{R}$ such that (2) is true for all ϕ. Thus δ is a singular dist^n.

A slight extension is $(\forall \phi)$

$$\langle \delta_\xi, \phi \rangle \equiv \phi(\xi) = \int_a^b \delta(x-\xi)\phi(x)\,dx$$

with alternate notations

$$\langle \delta(x-\xi), \phi(x) \rangle = \phi(\xi)$$

It is easily verified that δ is linear and $\langle \delta, \phi_n \rangle \to 0$ if $\phi_n \to 0$.

EXAMPLE Expected value distribution

Random variables have probability distributions which define the likelyhood of an event.

If $\rho(x)$ is the probability density function for a random variable $\mathbb{X}$ then

$$P(a < \mathbb{X} < b) = \int_a^b \rho(x)\,dx$$

is the probability that $\mathbb{X} \in (a, b)$. Then

$$\text{(1)} \qquad \langle E, \phi \rangle \equiv \int_a^b \rho(x)\,\phi(x)\,dx$$

defines a regular distribution.

In statistics the integral in (1) is the expected value of ϕ an is sometimes written

$$E(\phi) = \langle E, \phi \rangle$$

If $\phi(x) = x$ then $E(\phi)$ is the mean value of $\mathbb{X}$

Example Principal Value Distributions

Certain undefined, ambiguously defined or divergent integrals can be well defined as distributions.

For example (integration on $(-1,1)$ wlog)

$$(1)\quad \lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{1}{x}\,dx = \lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon}\frac{1}{x}\,dx + \int_{\varepsilon}^{1}\frac{1}{x}\,dx\right)$$

$$= \lim_{\varepsilon\to 0}\left(\ln|x|\Big|_{-1}^{-\varepsilon} + \ln|x|\Big|_{\varepsilon}^{1}\right)$$

$$= 0$$

whereas

$$(2)\quad \lim_{\varepsilon\to 0^+}\left(\int_{-1}^{-\varepsilon}\frac{dx}{x} + \int_{2\varepsilon}^{1}\frac{dx}{x}\right) = -\ln 2$$

The former is used to define the Principal Value dist^n $\frac{1}{x}$:

$$\left\langle \frac{1}{x}, \phi \right\rangle \equiv \lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{\phi(x)}{x}\,dx = \fint \frac{\phi(x)}{x}\,dx$$

The latter is common alternative not^n. In (1) we showed

$$\fint \frac{1}{x}\,dx = 0$$

This dist^n relates to the Fourier Trans of $H(x)$.

Distribution Properties

Algebraic and calculus properties of singular dist^{n}s are defined to match those of regular distributions

<u>Defn</u>: Let $T(x)$ be a distribution on D, $\alpha \in \mathbb{R}$ and $f(x)$ be some locally integrable function.

Then the distributions $T'(x)$, $T(x/\alpha)$, $T(x-\alpha)$ and $f(x)T(x)$ are defined by

(1) $\langle T'(x), \phi \rangle \equiv -\langle T(x), \phi'(x) \rangle$

(2) $\langle T(\frac{x}{\alpha}), \phi \rangle \equiv \langle T(x), \alpha\phi(\alpha x) \rangle$

(3) $\langle T(x-\alpha), \phi \rangle \equiv \langle T(x), \phi(\alpha + x) \rangle$

(4) $\langle f(x)T(x), \phi \rangle \equiv \langle T(x), f(x)\phi(x) \rangle$

So, for example, if $T(x)$ were regular ($D = C_0^\infty(\mathbb{R})$)

$$\langle T'(x), \phi \rangle = \int_{\mathbb{R}} T'(x)\phi(x)dx$$

$$= T(x)\overset{0}{\cancel{\phi(x)}}\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} T(x)\phi'(x)dx$$

$$= -\langle T(x), \phi(x) \rangle$$

Demonstrates property (1).

To show property (2) for regular dist^{n}s

$$\langle T(\tfrac{x}{a}), \phi \rangle = \int_{\mathbb{R}} T(\tfrac{x}{a})\, \phi(x)\, dx \qquad u = \frac{x}{a}$$

$$= \int_{\mathbb{R}} T(u)\; a\, \phi(au)\, du$$

$$= \langle T(x), a\phi(ax) \rangle$$

Property (3) is similar and (4) follows from

$$\langle f(x)T(x), \phi \rangle = \int_{\mathbb{R}} T(x)\left(f(x)\phi(x)\right) dx = \langle T, f\phi \rangle$$

<u>EXAMPLES</u> $\delta(x)$ dist^n, $\langle \delta(x), \phi \rangle \equiv \phi(0)$

$\delta'(x)$ is the dist^n defined by

$$\langle \delta'(x), \phi \rangle = -\phi'(0)$$

Also

$$\langle \delta(x-7), \phi \rangle \equiv \langle \delta(x), \phi(x+7) \rangle = \phi(7)$$

And if $T(x) = 2\delta'(x) + \delta\left(\frac{x+1}{2}\right)$

$$\langle T(x), \phi \rangle = 2\langle \delta'(x), \phi \rangle + \langle \delta\left(\tfrac{x+1}{2}\right), \phi \rangle$$

$$= -2\phi'(0) + \langle \delta(x), 2\phi(2x-1) \rangle$$

$$= -2\phi'(0) + 2\phi(-1)$$

Lastly $\langle \cos x\, \delta(x), \phi \rangle = \langle \delta(x), \cos x\, \phi \rangle = \phi(0)$
so that $\cos x\, \delta(x) = \delta(x)$ in the distributional sense.

<u>Example</u> The Heaviside function $H(x)$ defines a regular dist^n $(D = C_0^\infty(\mathbb{R}))$

$$\langle H, \phi \rangle = \int_0^\infty \phi(x)\,dx$$

We prove that $H'(x) = \delta(x)$ in the distributional sense.

By defn the distributional derivative $H'(x)$ must satisfy

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle \qquad \forall \phi \in D$$

$$= -\int_{\mathbb{R}} H(x)\phi'(x)\,dx$$

$$= -\int_0^\infty \phi'(x)\,dx$$

$$= \phi(0) - \cancel{\lim_{x\to\infty} \phi(x)}^{\,0} \qquad \text{compact support of } \phi$$

$$= \phi(0)$$

$$= \langle \delta(x), \phi \rangle$$

Thus $\langle H'(x), \phi \rangle = \langle \delta(x), \phi \rangle \quad \forall \phi \Rightarrow$

$$H'(x) = \delta(x)$$

in the distributional sense.

Delta Sequences

Defn A delta sequence $\{\delta_n(x)\}$ is a sequence of functions such that

$$\lim_{n \to \infty} \int_a^b \delta_n(x)\,\phi(x)\,dx = \phi(0)$$

for all $\phi \in D$.

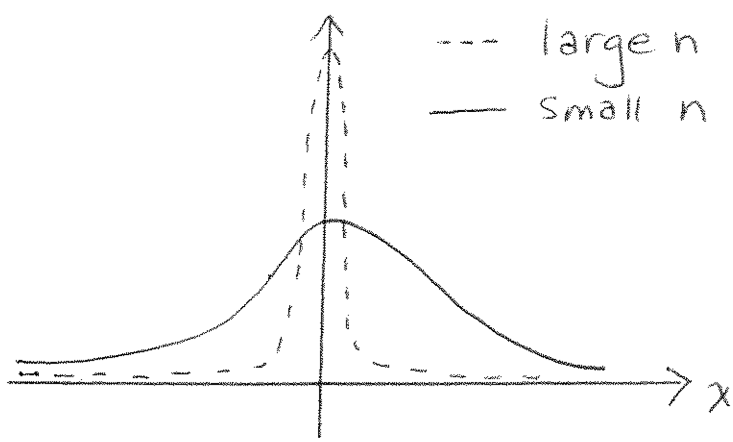
Remarks: For each n, $\langle \delta_n, \phi \rangle$ is a regular distⁿ. The defn states that $\langle \delta_n, \phi \rangle \to \langle \delta, \phi \rangle = \phi(0)$. This illustrates how a sequence of regular distⁿs can converge to a singular one.

We now list some examples (without proof)

EXAMPLE For $D = C_0^\infty(\mathbb{R})$

$$\delta_n(x) = \frac{1}{\pi} \frac{n}{(1 + n^2 x^2)}$$

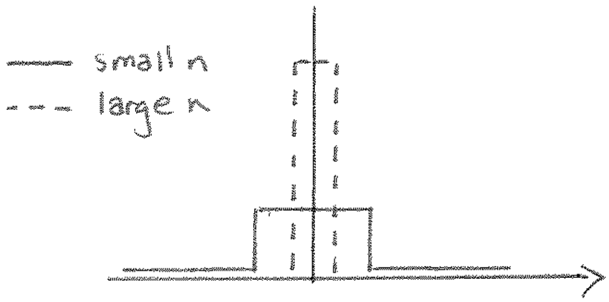
$$\int_{\mathbb{R}} \delta_n(x)\,dx = 1, \ \forall n$$



EXAMPLE $\quad D = C_0^\infty(\mathbb{R})$

$$\delta_n(x) = \begin{cases} n & |x| \le \frac{1}{2n} \\ 0 & |x| > \frac{1}{2n} \end{cases} \quad \Rightarrow \quad \int_{\mathbb{R}} \delta_n(x)\,dx = 1$$

where the graph is



Like the previous sequence

$$\lim_{n \to 0} \delta_n(x) = 0 \quad \text{if } x \neq 0$$

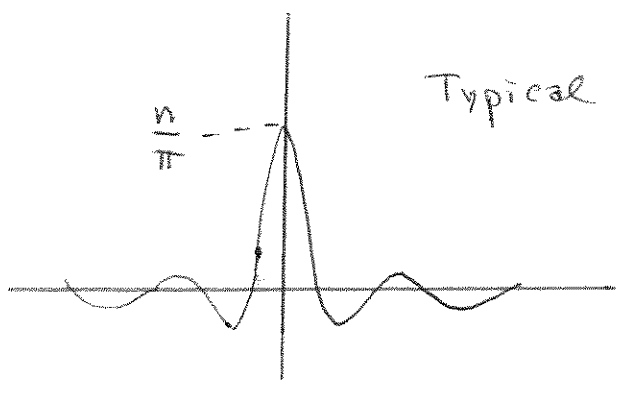
EXAMPLE

$$\delta_n(x) = \frac{\sin(nx)}{\pi x} \quad \text{for } x \neq 0 \text{ (removable discont}^{y}\text{)}$$

Is a delta sequence as well. One can see (L'Hospital rule) that $\delta_n \to \frac{n}{\pi}$ as $x \to 0$. Harder to prove $\delta_n(x) \to 0$ as $n \to \infty$. Still harder is the integral of δ_n over $\mathbb{R}$ is one.

$$\int_{\mathbb{R}} \delta_n(x)\,dx = 1$$

(Complex Var. Theory)



Typical δ_n

Distributional Solutions to Differential Eqns

Defn The distribution u on D is a distributional solution of

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = f$$

if

$$(1) \qquad \langle Lu, \phi \rangle = \langle f, \phi \rangle \qquad \forall \phi \in D$$

Here $a_K(x)$ are integrable functions and f may be a distribution or integrable function

whether u is a distribution or a function

$$\begin{aligned}
\langle Lu, \phi \rangle &= \langle a_2u'' + a_1u' + a_0u, \phi \rangle \\
&= \langle u'', a_2\phi \rangle + \langle u', a_1\phi \rangle + \langle u, a_0\phi \rangle \\
&= \langle u, (a_2\phi)'' \rangle - \langle u, (a_1\phi)' \rangle + \langle u, a_0\phi \rangle \\
&= \langle u, \underbrace{(a_2\phi)'' - (a_1\phi)' + a_0\phi}_{\text{Define to be } L^*\phi} \rangle
\end{aligned}$$

Thus, we define the <u>formal adjoint operator</u>

$$L^*\phi \equiv (a_2\phi)'' - (a_1\phi)' + a_0\phi$$

Then, u is a distributional solution of

$$Lu = f$$

only if

$$\boxed{(2) \qquad \langle u, L^*\phi \rangle = \langle f, \phi \rangle, \quad \forall \phi \in D}$$

If f is the delta function distributional solutions get a special name

<u>Defn</u> A <u>fundamental solution</u> is a distributional solution of

$$Lu = \delta(x-\xi)$$

(Such solns are not unique)

<u>Defn</u> The operator L is <u>formally self adjoint</u> if

$$L = L^*$$

in the sense $L\phi = L^*\phi$, $\forall \phi \in D$. This definition does not address operator domains

EXAMPLE Compute L^*

$$Lu \equiv xu'' + xu' + u$$

Then

$$L^*u = (xu)'' - (xu)' + u$$

$$L^*u = (xu' + u)' - (xu' + u) + u$$

$$L^*u = xu'' + (2-x)u'$$

Clearly $Lu \neq L^*u$ for all $u \in D$ so L is not formerly self adjoint

EXAMPLE Sturm Liouville Operators $(p \in C^2[a,b])$

$$Lu \equiv -(p(x)u')' + q(x)u$$

Compute the adjoint from

$$Lu = -pu'' - p'u' + qu$$

$$L^*u = -(pu)'' - (p'u)' + qu$$

$$= -(pu' + p'u)' - (p''u + p'u') + qu$$

$$= -(pu'' + \cancel{2}p'u' + \cancel{p''u}) - (\cancel{p''u} + \cancel{p'u'}) + qu$$

$$= -pu'' - p'u' + qu$$

Hence $Lu = L^*u$, $\forall u \in D$ and L is formally self adjoint.

Greens functions (Sturm Liouville)

$$Lu \equiv -(pu')' + qu$$

where $p(x), q(x)$ are sufficiently smooth functions. A distributional solution $g(x,\xi)$ of

$$(1) \qquad Lg = \delta(x-\xi)$$

must satisfy

$$\langle Lg, \phi \rangle = \langle \delta(x-\xi), \phi \rangle \qquad \forall \phi$$

$$\langle Lg, \phi \rangle = \phi(\xi)$$

$$\langle g, L\phi \rangle = \phi(\xi) \qquad \forall \phi \in D$$

Thus if $L\phi = f$ for some function $f(x)$ we have

$$(2) \qquad \phi(\xi) = \langle g, f \rangle$$

That is to say if g is a <u>regular</u> distributional soln of (1) then

$$(3) \qquad \phi(\xi) = \int_a^b g(x,\xi)\, f(x)\, dx$$

is a solution of

$$(4) \qquad L\phi = f$$

Eqns (3)-(4) show g is a Green's fn.

EXAMPLE Greens Function for $Lu = -u''$

Previously we showed that

$$(1) \qquad g(x,\xi) = \begin{cases} \xi(1-x) & \xi < x \\ x(1-\xi) & \xi > x \end{cases}$$

was the Green's function for the problem

$$(2) \qquad Lu \equiv -u'' = f(x) \qquad u(0) = u(1) = 0$$

The soln of (2) can be expressed as a regular distribution, i.e.

$$u(x) = \int_0^1 g(x,\xi)\, f(\xi)\, d\xi$$

Now we show that $g(x,\xi)$ is also the distributional solution of

$$(3) \qquad Lg = \delta(x-\xi)$$

$$(4) \qquad g(0,\xi) = g(1,\xi) = 0$$

Clear (1) implies (4) is satisfied. Sufficies now to show

$$\langle g, L^*\phi \rangle = \phi(\xi) \qquad \forall \phi \in D.$$

$$\langle g, -\phi'' \rangle = \phi(\xi)$$

Toward this end

$$\langle g, -\phi'' \rangle = -\int_0^1 g(x,\xi)\,\phi''(\xi)\,d\xi$$

$$\langle g, -\phi'' \rangle = -\int_0^x \xi(1-x)\phi''\,d\xi \;-\int_x^1 x(1-\xi)\phi''(\xi)\,d\xi$$

$$\langle g, -\phi'' \rangle = -(1-x)\underbrace{\int_0^x \xi\phi''(\xi)\,d\xi}_{H_1(x)} \;- x\underbrace{\int_x^1 (1-\xi)\phi''(\xi)\,d\xi}_{H_2(x)}$$

To simplify this we note that by integrating by parts twice (using $\phi(0)=\phi(1)=0$)

$$H_1(x) = \int_0^x \xi\phi''(\xi)\,d\xi \;=\; x\phi'(x) - \phi(x)$$

$$H_2(x) = \int_x^1 (1-\xi)\phi''(\xi)\,d\xi \;=\; -(1-x)\phi'(x) - \phi(x)$$

Using these in $\langle g, -\phi'' \rangle$ above and simplifying

$$\langle g, -\phi'' \rangle = \phi(x)$$