Eigenvectors, spaces and values

$$Ax = \lambda x \qquad A \in \mathbb{C}^{n\times n}$$

nontrivial solns only for certain λ:

$$p(\lambda) \equiv \det(A - \lambda I) = 0$$

For each eigenvalue $\lambda_k \ni p(\lambda_k) = 0$ we define its associated eigenspace

$$E_{\lambda_k}(A) \equiv \{x : Ax = \lambda_k x\}$$

is a vector space.

$$\dim E_{\lambda_k}(A) = \text{geometric multiplicity}$$

EXAMPLE

$$A \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad p(\lambda) = \lambda^2(1-\lambda)$$

λ	Alg mult.	Geom. mult.
$\lambda_1 = 0$	2	1
$\lambda_2 = 1$	1	1

Specifically

$$E_{\lambda_1}(A) = \text{span}\{\vec{\xi}_1\} \qquad \vec{\xi}_1 = (0,1,0)^T$$

$$E_{\lambda_2}(A) = \text{span}\{\vec{\xi}_2\} \qquad \vec{\xi}_2 = (1,0,0)^T$$

Note $\vec{\xi}_k$ don't span $\mathbb{R}^3$, i.e., $\mathbb{R}^3 \neq E_{\lambda_1}(A) \oplus E_{\lambda_2}(A)$

Basic Theorems on diagonalization

1) λ_k of A distinct $\Rightarrow$ A has n-ind eigvects $\vec{\xi}_1, \ldots \vec{\xi}_n$

2) If A has n-ind eigvects $\vec{\xi}_1 \ldots \vec{\xi}_n$ $\Rightarrow$ $\exists C$ such that $\Lambda = C^{-1}AC = \mathrm{diag}(\lambda_1, \ldots \lambda_n)$

Proof of 1) Let $(\lambda_k, \vec{\xi}_k)$ be eigenvalue/eigenvect pairs. Proof by induction

$$T_1 = \{\vec{\xi}_1\} \qquad \text{ind. set}$$

now assume following independent:

$$T_{k-1} = \{\vec{\xi}_1, \ldots \vec{\xi}_{k-1}\}$$

Want to show $\vec{\xi}_k$ ind of set T_{k-1}.

$$(1) \qquad \alpha_1 \vec{\xi}_1 + \cdots \alpha_k \vec{\xi}_k = \vec{0}$$

Consider $(A - \lambda_k I)$ acting on (1)

$$\alpha_1 (\lambda_1 - \lambda_k) \vec{\xi}_1 + \cdots \cdots \alpha_{k-1} (\lambda_{k-1} - \lambda_k) \vec{\xi}_{k-1} = \vec{0}$$

where in particular $(A - \lambda_k I)\vec{\xi}_k = \vec{0}$. But $\vec{\xi}_1 \ldots \vec{\xi}_{k-1}$ are independent and λ_k distinct hence $\alpha_1 = \cdots \alpha_{k-1} = 0$ and $\alpha_k = 0$ from (1) $\square$

Proof of 2)

$$AC = \left[A\vec{\xi}_1 \vdots \quad \vdots A\vec{\xi}_n \right] = \Lambda C$$

Adjoint and Self adjoint matrices

Define Euclidean inner product on $S = \mathbb{C}^n$

$$\langle x, y \rangle \equiv \sum_{j=1}^{n} x_j \bar{y}_j$$

Then the adjoint A^* of $A \in \mathbb{C}^{n\times n}$ is that matrix such that

$$(1) \quad \boxed{\langle Ax, y \rangle = \langle x, A^* y \rangle \qquad \forall x, y \in \mathbb{C}^n}$$

Rewrite LHS of (1) for $A = [a_{ij}]$

$$\langle Ax, y \rangle = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \bar{y}_i$$

$$= \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{n} a_{ij} \bar{y}_i \right)}_{A^T \bar{y}} x_j$$

From which we conclude

$$\boxed{A^* = \overline{A^T}}$$

Two subcases

$A \in \mathbb{R}^{n\times n}$, $A = A^T$ $\Rightarrow$ self adjoint / symmetric

$A \in \mathbb{C}^{n\times n}$, $A = \overline{A^T}$ $\Rightarrow$ self. adj. / Hermitian

Theorem 1 Let $A = A^* \in \mathbb{C}^{n \times n}$

1) $\langle Ax, x \rangle \in \mathbb{R}$
2) eigenvalues of A are real
3) $\lambda_1 \neq \lambda_2 \Rightarrow \vec{\xi}_1 \perp \vec{\xi}_2$
4) $\Pi = \{\vec{\xi}_1, \dots \vec{\xi}_n\}$ is a basis for $\mathbb{C}^n$
5) $\exists$ a unitary Q with $Q^{-1} = Q^*$ s.t.

$$Q^* A Q = \Lambda = \mathrm{diag}(\lambda_1 \dots \lambda_n)$$

Pf 1) $\langle Ax, x \rangle = \langle x, A^* x \rangle = \overline{\langle A^* x, x \rangle} = \overline{\langle Ax, x \rangle}$

Pf 2) Let $A\vec{\xi} = \lambda \vec{\xi}$. Then $\langle A\xi, \xi \rangle = \lambda \|\xi\|^2$

Pf 3) Let $\lambda_1 \neq \lambda_2$ and $A\vec{\xi}_K = \lambda_K \vec{\xi}_K$.

$$\begin{aligned}
\lambda_1 \langle \vec{\xi}_1, \vec{\xi}_2 \rangle &= \langle \lambda_1 \vec{\xi}_1, \vec{\xi}_2 \rangle \\
&= \langle A\vec{\xi}_1, \vec{\xi}_2 \rangle \\
&= \langle \vec{\xi}_1, A^* \vec{\xi}_2 \rangle \quad A = A^* \\
&= \langle \vec{\xi}_1, A \vec{\xi}_2 \rangle \\
&= \langle \vec{\xi}_1, \lambda_2 \vec{\xi}_2 \rangle \quad \lambda_K \in \mathbb{R} \\
&= \lambda_2 \langle \vec{\xi}_1, \vec{\xi}_2 \rangle
\end{aligned}$$

From this we deduce

$$(\lambda_1 - \lambda_2) \langle \vec{\xi}_1, \vec{\xi}_2 \rangle = 0$$

Since $\lambda_1 \neq \lambda_2$

$$\langle \vec{\xi}_1, \vec{\xi}_2 \rangle = 0$$

To prove 4) and then 5) we need to appeal to Theorem 2 below

Theorem 2 Let $M \subset \mathbb{C}^n$ be a linear invariant manifold for $A \in \mathbb{C}^{n \times n}$

(a) $x, y \in M \Rightarrow \alpha x + \beta y \in M$ closure.

(b) $x \in M \Rightarrow Ax \in M$

Then $\exists (\lambda, x) \in \mathbb{C} \times M$ such that $Ax = \lambda x$

<u>Remark:</u> M contains at least one eigenvector

<u>Proof</u> M has a basis $T = \{\phi_1, \dots \phi_K\}$, $k \le n$. Let

$$x = \sum_{i=1}^{K} \alpha_i \phi_i \qquad A\phi_i = \sum_{j=1}^{K} \beta_{ij} \phi_j$$

The latter requires invariance, $A\phi_i \in M$. Then

$$Ax = \sum_{i=1}^{K} \alpha_i A\phi_i = \sum_{i=1}^{K} \sum_{j=1}^{K} \alpha_i \beta_{ij} \phi_j$$

Want to show $\exists x \in M$, $x \neq 0$ s.t $Ax - \lambda x = 0$.

$$\sum_{i=1}^{K} \alpha_i \left(\sum_{j=1}^{K} (\beta_{ij} - \lambda \delta_{ij}) \phi_j \right) = 0 \qquad \delta_{ij} \text{ Kronecker}$$

Since $\{\phi_j\}$ ind, coefficients of ϕ_j must vanish

$$\sum_{i=1}^{K} (\beta_{ij} - \lambda \delta_{ij}) \alpha_i = 0 \iff (B - \lambda I)\vec{\alpha} = \vec{0}$$

But every B has at least one e-value. □

And now to complete Theorem 1 regarding selfadjoint matrices.

Pf 4) The matrix A has at least one e-vector pair $(\lambda_1, \vec{\xi}_1)$, $A\vec{\xi}_1 = \lambda_1 \vec{\xi}_1$.

$$T_1 = \{\vec{\xi}_1\}$$

is an 'orthogonal' set. Induction. Let

$$T_{k-1} = \{\vec{\xi}_1, \ldots \vec{\xi}_{k-1}\}$$

be an orthogonal set of e-vectors. Define orthogonal complement space

$$M_k \equiv \{x : \langle x, \vec{\xi}_i \rangle = 0 \quad \forall i = 1, \ldots k-1\}$$

M_k is a linear invariant manifold for A

$$\langle Ax, \vec{\xi}_i \rangle = \langle x, A\vec{\xi}_i \rangle = \lambda_i \langle x, \vec{\xi}_i \rangle = 0$$

By Theoreom 2 there exists a pair $(\lambda_k, \vec{\xi}_k) \in \mathbb{R} \times M_k$ such that

$$A\vec{\xi}_k = \lambda \vec{\xi}_k \qquad \vec{\xi}_k \neq 0$$

and by defn of M_k, $\vec{\xi}_k \notin \operatorname{span} T_{k-1}$. Add $\vec{\xi}_k$ to get T_k and iterate. □

Pf 5) With no great detail, take T in above proof and apply Gram Schmidt to obtain orthonormal basis. The columns of Q are this orthonormal basis □

Diagonalization Example

$$A = A^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Theorem guarantees $\exists Q$ with $Q^{-1} = Q^T$ (orthog) that diagonalizes A symmetric, self adjoint.

$$p(\lambda) = \det(A - \lambda I) = \lambda(\lambda - 2)^2$$

For $\lambda_1 = 0$

$$A - \lambda_1 I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\vec{\xi}_1 = (0, -1, 1)^T$ is a spanning vector for $E_{\lambda_1}(A)$

For $\lambda_2 = 2$ it's easy to show

$$A - \lambda_2 I \sim \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} \vec{\xi}_2^{(1)} &= (0, 1, 1)^T \\ \vec{\xi}_2^{(2)} &= (1, 0, 0)^T \end{aligned}$$

Notice $\vec{\xi}_2^{(k)}$ are orthogonal. Had we picked a different pair one would use Gram Schmidt to find an orthogonal basis for $E_{\lambda_2}(A)$

Normalized $\vec{\xi}_k$ are columns of Q

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Then

$$A = Q \Lambda Q^T \qquad \Lambda = \mathrm{diag}(0, 2, 2)$$

Orthogonal Diagonalization Remarks

Suppose $A = A^T$ and

$$A = Q \Lambda Q^{-1} \qquad\qquad Q^{-1} = Q^T$$

This defines a similarity transformation.

As a result we have the following commutation diagram

$$\boxed{\begin{array}{l} x = Qy \\ b = Qb' \end{array}}$$

$$\begin{tikzcd} Ax = b \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & x = A^{-1} b \arrow[d, leftrightarrow] \\ \Lambda y = b' \arrow[r, leftrightarrow] & y = \Lambda^{-1} b' \end{tikzcd}$$

From a theoretical point $Ax = b$ is easy to invert once Q is found. $Q^{-1} = Q^T$ so inversion of Q is trivial.

One could (for large matrices) first solve

$$\Lambda y = b' \qquad\qquad \Lambda \text{ diagonal}$$

then the soln is

$$x = Qy$$

which has $O(n^2)$ operations for $A \in \mathbb{R}^{n \times n}$

Theorem (Maximum Principle) If $A = A^T$ then

$$\lambda_1 = \max_{\|x\|=1} \langle Ax, x \rangle$$

where λ_1 is the largest e-val of A

Pf/ Let Q diagonalize A

$$A = Q \Lambda Q^T \qquad Q = [\vec{\xi}_1, \ldots \vec{\xi}_n] \quad \Lambda = \mathrm{diag}(\lambda_1 \ldots \lambda_n)$$

where $\lambda_1 \geq \lambda_2 \ldots \lambda_n$ and $T = \{\phi_k\}_{k=1}^n$ is an orthonormal set.

$$\begin{aligned} \langle Ax, x \rangle &= (Ax)^T x \\ &= x^T A x \\ &= x^T Q \Lambda Q^T x \\ &= y^T \Lambda y \qquad \text{where } x \equiv Q y \end{aligned}$$

Now note that $\|y\| = \|Qx\| = \|x\| = 1$ since orthogonal matrices preserve length. Thus

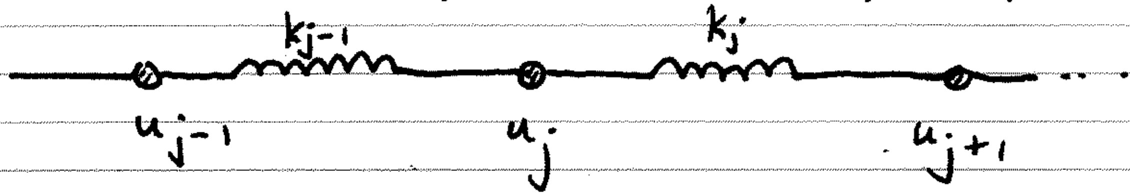
$$\begin{aligned} \langle Ax, x \rangle &= \lambda_1 y_1^2 + \cdots \lambda_n y_n^2 \qquad y = (y_1 \ldots y_n)^T \\ &\leq \lambda_1 (y_1^2 + \cdots y_n^2) \\ &\leq \lambda_1 \end{aligned}$$

The maxima is realized with $y = (1, 0 \ldots 0)^T$ $\square$

Remark More general versions have constraints on x having to be orthogonal to

$$E_{k-1} \equiv E_{\lambda_1}(A) \oplus \cdots E_{\lambda_{k-1}}(A)$$

Lattice Vibrations (application max principle)



Equal masses m; spring constants k_j unequal

$$m \frac{d^2 u_j}{dt^2} = k_{j-1} \underbrace{(u_{j-1} - u_j)}_{\text{spring extension}} + k_j (u_{j+1} - u_j)$$

Can be written in matrix form

$$\frac{d^2 u}{dt^2} = A u \qquad u = (u_1, \ldots u_n)^T$$

and A is tridiagonal.

What can we say about periodic motion?
Suppose

$$(1) \qquad u = e^{i\omega t} x \qquad x \in \mathbb{R}^n$$

Then

$$A x = -\omega^2 x$$

is an e-val problem for ω.

Here (1) presumes there is synchronous periodic soln.

Use max principle to make bounds on such ω.

$$\langle Au, u\rangle = \sum_i \sum_j a_{ij} u_i u_j$$

$$(2) \quad \langle Au, u\rangle = \frac{1}{m}\left(-k_0 u_1^2 - k_n u_n^2 - \sum_{j=1}^{n-1} k_j (u_j - u_{j+1})^2\right)$$

The dominant e-value of A

$$\lambda_1 = \max_{\|u\|=1} \langle Au, u\rangle$$

From (2) we can prove:

$m \uparrow \quad \Rightarrow |\lambda_1| \downarrow \quad$ slower motion

$k_j \uparrow \quad \Rightarrow |\lambda_1| \uparrow \quad$ faster motion