Eigenvectors, spaces and values

\[ A \mathbf{x} = \lambda \mathbf{x} \quad A \in \mathbb{C}^{n \times n} \]

nontrivial solns only for certain \( \lambda \):

\[ p(\lambda) = \det (A - \lambda I) = 0 \]

For each eigenvalue \( \lambda_k \in p(\lambda_k) = 0 \) we define its associated eigenspace

\[ E_{\lambda_k}(A) = \{ \mathbf{x} : A \mathbf{x} = \lambda_k \mathbf{x} \} \]

is a vector space.

\[ \text{dim } E_{\lambda_k}(A) = \text{geometric multiplicity} \]

**Example**

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad p(\lambda) = \lambda^2 (1 - \lambda) \]

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Alg mult.</th>
<th>Geom. mult.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 = 0 )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_2 = 1 )</td>
<td>1</td>
<td>1</td>
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Specifically

\[ E_{\lambda_1}(A) = \text{span} \{ \mathbf{z}_1 \} \quad \mathbf{z}_1 = (0, 1, 0)^T \]

\[ E_{\lambda_2}(A) = \text{span} \{ \mathbf{z}_2 \} \quad \mathbf{z}_2 = (1, 0, 0)^T \]

Note \( \mathbf{z}_k \) don't span \( \mathbb{R}^3 \), i.e., \( \mathbb{R}^3 \neq E_{\lambda_1}(A) \oplus E_{\lambda_2}(A) \)
Basic Theorems on diagonalization

1) \( \lambda_k \) of \( A \) distinct \( \Rightarrow \) \( A \) has n-ind eigenvectors \( \frac{y_1}{\lambda_1}, \ldots, \frac{y_n}{\lambda_n} \)

2) If \( A \) has n-ind eigenvectors \( \frac{y_1}{\lambda_1}, \ldots, \frac{y_n}{\lambda_n} \) \( \Rightarrow \) \( \exists \mathbf{C} \) such that \( \mathbf{A} = \mathbf{C}^{-1}\mathbf{AC} = \text{diag}(\lambda_1, \ldots, \lambda_n) \)

Proof of 1) Let \( (\lambda_k, \frac{y_k}{\lambda_k}) \) be eigenvalue/eigenvector pairs. Proof by induction

\[ T_1 = \left\{ \frac{y_1}{\lambda_1} \right\} \text{ ind. set} \]

Now assume following independent:

\[ T_{k-1} = \left\{ \frac{y_1}{\lambda_1}, \ldots, \frac{y_{k-1}}{\lambda_{k-1}} \right\} \]

Want to show \( \frac{y_k}{\lambda_k} \) ind of set \( T_{k-1} \).

(1) \( \alpha_1 \frac{y_1}{\lambda_1} + \ldots + \alpha_k \frac{y_k}{\lambda_k} = 0 \)

Consider \( (A - \lambda_k I) \) acting on (1)

\[ \alpha_1 (\lambda_1 - \lambda_k) \frac{y_1}{\lambda_1} + \ldots + \alpha_k (\lambda_{k-1} - \lambda_k) \frac{y_k}{\lambda_{k-1}} = 0 \]

where in particular \( (A - \lambda_k I) \frac{y_k}{\lambda_k} = 0 \). But \( \frac{y_1}{\lambda_1}, \ldots, \frac{y_{k-1}}{\lambda_{k-1}} \) are independent and \( \lambda_k \) distinct hence \( \alpha_k = \cdots = \alpha_{k-1} = 0 \) and \( \frac{y_k}{\lambda_k} \) independent from (1) \( \square \)

Proof of 2)

\[ AC = \begin{bmatrix} A \frac{y_1}{\lambda_1} & \ldots & A \frac{y_n}{\lambda_n} \end{bmatrix} = \Lambda \mathbf{C} \]
Adjoint and Self-adjoint matrices

Define Euclidean inner product on $S = \mathbb{C}^n$

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$$

Then the adjoint $A^*$ of $A \in \mathbb{C}^{n \times n}$ is the matrix such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall x, y \in \mathbb{C}^n \quad (1)$$

Rewrite LHS of (1) for $A = [a_{ij}]$

$$\langle Ax, y \rangle = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} x_j \overline{y_i} \right)$$

$$= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} a_{ij} \overline{y_i} \right) x_j$$

From which we conclude

$$A^* = \overline{A^T}$$

Two subcases

$A \in \mathbb{R}^{n \times n}$, $A = A^T \Rightarrow$ self-adjoint/symmetric

$A \in \mathbb{C}^{n \times n}$, $A = \overline{A^T} \Rightarrow$ self-adjoint/Hermitian
Theorem 1. Let $A = A^* \in \mathbb{C}^{n \times n}$

1) $\langle Ax, x \rangle \in \mathbb{R}$
2) eigenvalues of $A$ are real
3) $\lambda_1 \neq \lambda_2 \Rightarrow \vec{z}_1 \perp \vec{z}_2$
4) $\vec{z}_1, \ldots, \vec{z}_n$ is a basis for $\mathbb{C}^n$
5) There exists a unitary $Q$ with $Q^{-1} = Q^* \text{s.t.}$

$$Q^*AQ = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

Proof 1. $\langle Ax, x \rangle = \langle x, A^*x \rangle = \langle A^*x, x \rangle = \langle Ax, x \rangle$

Proof 2. Let $A\vec{z} = \lambda\vec{z}$. Then $\langle A\vec{z}, \vec{z} \rangle = \lambda \| \vec{z} \|^2$

Proof 3. Let $\lambda_1 \neq \lambda_2$ and $A\vec{z}_k = \lambda_k \vec{z}_k$.

$$\lambda_1 \langle \vec{z}_1, \vec{z}_2 \rangle = \langle \lambda_1 \vec{z}_1, \vec{z}_2 \rangle = \langle A\vec{z}_1, \vec{z}_2 \rangle = \langle \vec{z}_1, A^*\vec{z}_2 \rangle = \langle \vec{z}_1, \vec{z}_2 \rangle$$

For all $A = A^*$

$$\langle \vec{z}_1, \vec{z}_2 \rangle = \langle \lambda_2 \vec{z}_1, \lambda_2 \vec{z}_2 \rangle \quad \lambda_2 \in \mathbb{R}$$

From this we deduce

$$\lambda_1 \lambda_2 \langle \vec{z}_1, \vec{z}_2 \rangle = 0$$

Since $\lambda_1 \neq \lambda_2$

$$\langle \vec{z}_1, \vec{z}_2 \rangle = 0$$
To prove 4) and then 5) we need to appeal to Theorem 2 below.

**Theorem 2** Let $M \subseteq \mathbb{C}^n$ be a linear invariant manifold for $A \in \mathbb{C}^{n \times n}$

(a) $x, y \in M \Rightarrow ax + by \in M$ closure.

(b) $x \in M \Rightarrow Ax \in M$

Then $\exists (\lambda, x) \in \mathbb{C} \times M$ such that $Ax = \lambda x$

Remark: $M$ contains at least one eigenvector.

Proof: $M$ has a basis $T = \{\phi_1, \ldots, \phi_k\}$, $k \leq n$. Let

$$x = \sum_{i=1}^{k} a_i \phi_i \quad A \phi_i = \sum_{j=1}^{k} \beta_{ij} \phi_j$$

The latter requires invariance, $A \phi_i \in M$. Then

$$Ax = \sum_{i=1}^{k} a_i A \phi_i = \sum_{i=1}^{k} \sum_{j=1}^{k} a_i \beta_{ij} \phi_j$$

Want to show $\exists x \in M, x \neq 0$ s.t $Ax - \lambda x = 0$.

$$\sum_{i=1}^{k} a_i \left( \sum_{j=1}^{k} (\beta_{ij} - \lambda \delta_{ij}) \phi_j \right) = 0 \quad \delta_{ij} \text{ kronecker}$$

Since $\delta_{ij}$ ind, coefficients of $\phi_j$ must vanish

$$\sum_{i=1}^{k} (\beta_{ij} - \lambda \delta_{ij}) a_i = 0 \Leftrightarrow (B - \lambda I) \mathbf{a} = 0$$

But every $B$ has at least one e-value. $QED$
And now to complete Theorem 1 regarding self-adjoint matrices.

Pf 4) The matrix $A$ has at least one e-vector pair $(\lambda_i, \vec{z}_i)$, $A \vec{z}_i = \lambda_i \vec{z}_i$.

$$T_i = \{ \vec{z}_i \}$$

is an 'orthogonal' set. Induction. Let

$$T_k = \{ \vec{z}_1, \ldots, \vec{z}_{k-1} \}$$

be an orthogonal set of e-vectors. Define orthogonal complement space

$$M_k = \{ x : \langle x, \vec{z}_i \rangle = 0 \ \forall i = 1, \ldots, k-1 \}$$

$M_k$ is a linear invariant manifold for $A$.

$$\langle Ax, \vec{z}_i \rangle = \langle x, A\vec{z}_i \rangle = \lambda_i \langle x, \vec{z}_i \rangle = 0$$

By Theorem 2 there exists a pair $(\lambda_k, \vec{z}_k) \in \mathbb{R} \times M_k$ such that

$$A \vec{z}_k = 2 \vec{z}_k, \quad \vec{z}_k \neq 0$$

and by defn of $M_k$, $\vec{z}_k \notin \text{span } T_k$. Add $\vec{z}_k$ to get $T_{k+1}$ and iterate. □

Pf 5) With no great detail, take $T$ in above proof and apply Gram-Schmidt to obtain orthonormal basis. The columns of $Q$ are this orthonormal basis. □
Diagonalization Example

\[ A = A^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

Theorem guarantees \( \exists Q \) with \( Q^{-1} = Q^T \) (orthog) that diagonalizes \( A \) symmetric, self-adjoint.

\[ p(\lambda) = \det(A - \lambda I) = \lambda (\lambda - 2)^2 \]

For \( \lambda_1 = 0 \)

\[ A - \lambda_1 I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

so \( \vec{z}_1 = (0, -1, 1)^T \) is a spanning vector for \( \text{E}_{\lambda_1}(A) \)

For \( \lambda_2 = 2 \) it's easy to show

\[ A - \lambda_2 I \sim \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{z}^{(1)}_2 = (0, 1, 1)^T \]

\[ \vec{z}^{(1)}_2 = (1, 0, 0)^T \]

Notice \( \vec{z}^{(k)} \) are orthogonal. Had we picked a different pair one would use Gram Schmidt to find an orthogonal basis for \( \text{E}_{\lambda_2}(A) \)

Normalized \( \vec{z}_k \) are columns of \( Q \)

\[ Q = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \]

Then

\[ A = Q \Lambda Q^T \]

\[ \Lambda = \text{diag}(0, 2, 2) \]
Orthogonal Diagonalization Remarks

Suppose $A = A^T$ and

$$A = Q \Lambda Q^{-1} \quad Q^{-1} = Q^T$$

This defines a similarity transformation.

As a result we have the following commutation diagram

$$Ax = b \quad \leftrightarrow \quad x = A^{-1}b$$

$$x = Qy$$
$$b = Qb'$$

$$\Lambda y = b' \quad \leftrightarrow \quad y = \Lambda^{-1}b'$$

From a theoretical point $Ax = b$ is easy to invert once $Q$ is found. $Q^{-1} = Q^T$ so inversion of $Q$ is trivial.

One could (for large matrices) first solve

$$\Lambda y = b'$$ \quad $\Lambda$ diagonal

then the soln is

$$x = Qy$$

which has $O(n^2)$ operations for $A \in \mathbb{R}^{n \times n}$
Theorem (Maximum Principle) If $A = A^T$ then

$$\lambda_1 = \max_{\|x\| = 1} \langle Ax, x \rangle$$

where $\lambda_1$ is the largest eigenvalue of $A$.

**Pf:** Let $Q$ diagonalize $A$

$$A = Q \Lambda Q^T, \quad Q = [z_1 \cdots z_n], \quad \Lambda = \text{diag}(\lambda_1 \cdots \lambda_n)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $T = \{z_k\}_{k=1}^n$ is an orthonormal set.

$$\langle Ax, x \rangle = (Ax)^T x$$

$$= x^T A x$$

$$= x^T Q \Lambda Q^T x$$

$$= y^T \Lambda y \quad \text{where} \quad x \equiv Q y$$

Now note that $\|y\| = \|Qx\| = \|x\| = 1$ since orthogonal matrices preserve length. Thus

$$\langle Ax, x \rangle = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \quad y = (y_1, \cdots, y_n)^T$$

$$\leq \lambda_1 \left(y_1^2 + \cdots + y_n^2\right)$$

$$\leq \lambda_1$$

The maxima is realized with $y = (1, 0, \ldots, 0)^T$.

**Remark:** More general versions have constraints on $x$ having to be orthogonal to

$$E_{k-1} \equiv E_{\lambda_1}(A) \oplus \cdots \oplus E_{\lambda_{k-1}}(A)$$
Lattice Vibrations (application max principle)

Equal masses \( m \); spring constants \( k_j \); unequal

\[
m \frac{d^2 u_j}{dt^2} = k_{j-1} (u_{j-1} - u_j) + k_j (u_{j+1} - u_j)
\]

Can be written in matrix form

\[
\frac{d^2 u}{dt^2} = A u \quad u = (u_1, \ldots, u_n)^T
\]

and \( A \) is tridiagonal.

What can we say about periodic motion?

Suppose

\[
(1) \quad u = e^{i\omega t} x \quad x \in \mathbb{R}^n
\]

Then

\[
A x = -\omega^2 x
\]

is an eigenvalue problem for \( \omega \).

Here (1) presumes there is synchronous periodic 

Here (1) presumes there is synchronous periodic behavior.
Use max principle to make bounds on such $u$.

\[ \langle Au, u \rangle = \sum_i \sum_j a_{ij} u_i u_j \]

(2) \[ \langle Au, u \rangle = \frac{1}{m} \left( -k_0 u_i^2 - k_n u_n^2 - \sum_{j=1}^{n-1} k_j (u_j - u_{j+1})^2 \right) \]

The dominant e-value of $A$

\[ \lambda_1 = \max \frac{\langle Au, u \rangle}{\|u\|_1^2} = 1 \]

From (2) we can prove:

\[ m \uparrow \Rightarrow |\lambda| \downarrow \quad \text{slower motion} \]

\[ k_j \uparrow \Rightarrow |\lambda_j| \uparrow \quad \text{faster motion} \]