Matrix Subspaces and Projections

Each $A \in \mathbb{C}^{n \times n}$ defines a linear operator:

(1) $\qquad L(x) \equiv Ax \qquad x \in \mathbb{C}^n$

Defn A linear manifold (subspace) $M \subset S$ is an invariant manifold of A if

$$x \in M \Rightarrow Ax \in M$$

Remarks: One could also state M is invariant under L since $L(M) \subset M$.

A simple example of an invariant manifold is an eigenspace $M = E_\lambda(A)$

Fundamental spaces

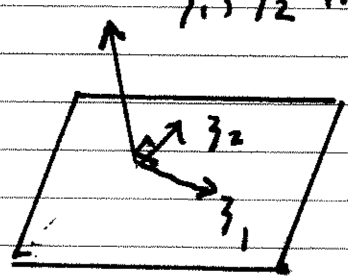
nullspace $\quad N(A) \equiv \{ x : Ax = 0 \}$

range $\quad R(A) \equiv \{ y : y = Ax \text{ for some } x \in \mathbb{C}^n \}$

orthog. $\quad M^\perp \equiv \{ x : x \in S, y \in M, \langle x, y \rangle = 0 \}$

EXAMPLE Let $S = \mathbb{R}^3$, $\langle \cdot, \cdot \rangle$ Euclid prod and ξ_1, ξ_2 independent

plane



$M^\perp = \text{span}\{ \vec{\xi}_1 \times \vec{\xi}_2 \}$

is line $\perp$ plane shown.

Defn Let $M \subset S$ be a linear manifold of S with orthonormal basis

$$T = \{\phi_1 \ldots \phi_m\} \quad , \quad \|\phi_k\| = 1$$

The orthogonal projection of $v \in S$ onto M is

$$v_\perp \equiv \sum_{i=1}^{m} \langle v, \phi_i \rangle \phi_i$$

From this we can further define the associated projection operator

$$P : S \to M$$

$$P(x) = \sum_{i=1}^{m} \langle v, \phi_i \rangle \phi_i$$

EXAMPLE $S = L^2(-1,1)$ and $M = \text{span}\{\phi_1, \phi_2\}$ where $\phi_1 = 1/\sqrt{2}$, $\phi_2 = \sqrt{\frac{3}{2}}\, x$.

Then for any $v = ax^2 + bx + c$

$$v_\perp = \langle v, \phi_1 \rangle \phi_1 + \langle v, \phi_2 \rangle \phi_2$$

$$v_\perp = \tfrac{1}{3}a + c + bx$$

where

$$\langle f, g \rangle \equiv \int_{-1}^{1} f(x)\, g(x)\, dx$$

Theorem Fredholm alternative $A \in \mathbb{C}^{m \times n}$

(1) $Ax = b$ has soln $\Leftrightarrow$ $\langle b, v \rangle = 0 \quad \forall v \in N(A^*)$

(2) $Ax = b$ has unique solution $\Leftrightarrow$ $N(A) = 0$

Proof of (1) $\Rightarrow$ Let $Ax = b$ and $v \in N(A^*)$

$$\langle v, b \rangle = \langle v, Ax \rangle = \langle A^* v, x \rangle = 0$$

Proof of (2) $\Leftarrow$ (Contradiction)

Assume $Ax = b$ has no soln but $\langle b, v \rangle = 0 \ \forall v \in N(A^*)$
Note then that $b \notin R(A)$*

Since $\mathbb{C}^n = R(A) \oplus R(A)^\perp$ one can write

$$b = b_r + b_\perp \qquad b_r \in R(A) \qquad b_\perp \in R(A)^\perp$$

Since $b_\perp \in R(A)^\perp$

$$\langle b_\perp, Ax \rangle = \langle A^* b_\perp, x \rangle = 0 \qquad \forall x$$

This implies $b_\perp \in N(A^*)$. By supposition
$\langle b, v \rangle = \langle b, b_\perp \rangle = 0$

$$\langle b, b_\perp \rangle = 0$$

$$\langle b_r + b_\perp, b_\perp \rangle = 0 \qquad (b_r \perp b_\perp)$$

$$\| b_\perp \|^2 = 0$$

Hence $b_\perp = 0$ and $b = b_r \in R(A)$ contradict* $\square$

Proof of (2) (contrapositive)

Let x be a solution of $Ax = b$ and assume $N(A) \neq 0$.

Thus $\exists y \neq 0$, $y \in N(A)$. For $\alpha \in \mathbb{R}$

$$\begin{aligned} A(x + \alpha y) &= Ax + \alpha Ay \\ &= b + 0 \\ &= b \end{aligned}$$

showing $x + \alpha y$ a soln too, i.e. not unique

Proof of (2) $\Leftarrow$

Assume $x_1 \neq x_2$ are solns of $Ax = b$.

$$A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0$$

So that $y = x_1 - x_2$ is a nonzero element of $N(A)$. Contradiction.

Remark

$$R(A) \perp N(A^*)$$

<u>EXAMPLE</u> For what α does $Ax = b$ have a solution where $b = (\alpha, 1, 1)^T$ and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \qquad \det A = 0$$

A simple calculation $N(A^*) = N(A^T) = \mathrm{span}\{(1,-1,-1)\}$

$$\langle b, v\rangle = (\alpha, 1, 1)(1,-1,-1)^T = \alpha - 2 = 0$$

By the FAT must have $\alpha = 2$. Soln not uniq.

<u>EXAMPLE</u> Prove $\langle Ax, x\rangle > 0 \ \forall x \neq 0 \Rightarrow A^{-1}$ exists

Let $v \in N(A^*)$. Then

$$\langle v, A^* v\rangle = \langle Av, v\rangle = 0$$

Since $\langle Av, v\rangle > 0 \ \forall v \neq 0$ conclude $v = 0$ and

$$N(A^*) = 0$$

By the FAT

$$Ax = b \text{ has soln} \Leftrightarrow \langle v, b\rangle = 0 \quad \forall v \in N(A^*)$$

is true, i.e. $Ax = b$ has soln $\forall b$.

Moreover soln unique since

$$Aw = 0 \Rightarrow \langle Aw, w\rangle = 0 \Leftrightarrow w = 0$$

so $N(A) = 0$ and soln unique.

EXAMPLE

$$(1) \quad F(\vec{x}, \varepsilon) = \begin{pmatrix} f_1(x,y,\varepsilon) \\ f_2(x,y,\varepsilon) \end{pmatrix} \equiv \begin{pmatrix} x - y - \varepsilon x^3 \\ x - y - \varepsilon y \end{pmatrix}$$

when $\varepsilon = 0$, $F(\vec{x}, 0) = A\vec{x} = 0$ for many $\vec{x}$.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \qquad N(A) = \text{span}\{(1,1)^T\}$$

Despite this the small perturbation $0 < \varepsilon \ll 1$ forces more particular answers. Such roots must be smooth in ε (IFThm)

$$(2) \quad x = \alpha + \varepsilon x_1 + O(\varepsilon^2)$$

$$(3) \quad y = \alpha + \varepsilon y_1 + O(\varepsilon^2)$$

Substituting these expansions yields $O(\varepsilon)$ problem

$$(4) \quad A\vec{x}_1 = \vec{b}_1 \qquad \vec{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} \alpha^3 \\ \alpha \end{pmatrix}$$

Problem (4) has a soln (a perturbation exists) only if $\langle \vec{v}, \vec{b}_1 \rangle \ \forall \vec{v} \in N(A^T) = N(A^*)$. Calculations show

$$N(A^T) = \text{span}\,\vec{v} \qquad \vec{v} = (1,-1)^T$$

FAT yields values of α:

$$\langle \vec{v}, \vec{b}_1 \rangle = \alpha^3 - \alpha = 0 \qquad \alpha = 0, \pm 1$$

Only for these α are (2)-(3) expansions for the roots of $F(\vec{x}, \varepsilon)$.

EXAMPLE Perturbed eigenvalue problems

$$A(\varepsilon)\,x(\varepsilon) = \lambda(\varepsilon)\,x(\varepsilon)$$

where

$$A(\varepsilon) = \begin{bmatrix} 1 & 2+\varepsilon \\ 2 & 1+\varepsilon \end{bmatrix} = A_0 + \varepsilon A_1$$

For the unperturbed $(\varepsilon = 0)$ matrix

$$\det(A - \lambda I) = (\lambda+1)(\lambda-3)$$

so that eigenvalues are $\lambda_0 = -1, 3$.

Expansion

$$(A_0 + \varepsilon A_1)(x_0 + \varepsilon x_1 + \cdots) = (\lambda_0 + \varepsilon\lambda_1 + \cdot)\,x(\varepsilon)$$

Collecting $O(1)$ and $O(\varepsilon)$ terms

$O(1)$ $\qquad A_0 x_0 = \lambda_0 x_0$

$O(\varepsilon)$ $\qquad (A_0 - \lambda_0 I)x_1 = \lambda_1 x_0 - A_1 x_0 = b$

Lets just consider $\lambda_0 = 3$ ($\lambda_0 = -1$ similar).

$$A_0 - \lambda_0 I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

which is symmetric and singular.

$$N(A_0 - \lambda_0 I) = \text{span}\,\vec{x}_0 \qquad \vec{x}_0 = (1,1)^T$$

$$N(A_0^T - \lambda_0 I) = \text{span}\,\vec{x}_0$$

Here $\vec{x}_0$ is the eigenvector for $\lambda_0 = 3$

By the Fred. Alt. Theorem

$$(A_0 - \lambda_0 I)\, x_1 = b \equiv \lambda_1 x_0 - A_1 x_0$$

has a solution only if

$$\langle x_0, b \rangle = 0 .$$

Given the defn of b :

$$\lambda_1 = \frac{\langle x_0, A_1 x_0 \rangle}{\| x_0 \|^2}$$

Calculations have (for $x_0 = (1,1)^T$) $\| x_0 \|^2 = 2$
and

$$A_1 x_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus

$$\lambda_1 = \frac{(1,1) \cdot (1,1)^T}{2} = 1$$

Conclude

$$\lambda(\varepsilon) = 3 + \varepsilon + O(\varepsilon^2)$$