Operator boundedness

Throughout $L: H \to H$ is linear on Hilbert space H.

Defn L is a bounded operator if $\exists M > 0$ s.t.

$$\|Lu\| \le M \|u\| \qquad \forall u \in H$$

Boundedness is akin to continuity:

Theorem (Stakgold for proof)

$$L \text{ continuous} \iff L \text{ bounded}$$

$$L \text{ continuous} \iff L\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} L z_n$$

EXAMPLE Let $k(x,y) \in C[a,b]^2$

$$Lu = \int_a^b k(x,y)\,u(y)\,dy \qquad \text{Bounded}$$

is a bounded operator on (all) $H = L^2[a,b]$

Let $v = Lu$

$$|v(x)|^2 = \left|\int_a^b k(x,y)\,u(y)\,dy\right|^2 \le \|u\|^2 \int_a^b |k(x,y)|^2\,dy$$

Holder inequality

Hence

$$\|Lu\|^2 \le \|u\|^2 \cdot \underbrace{\int_a^b\int_a^b |k(x,y)|^2\,dy\,dx}_{M}$$

Holder Inequalities

A set of inequalities for $L_p(\Omega)$ spaces.
For $1 < p, q < \infty$

$$f \in L^p(\Omega) \Leftrightarrow \|f\|_p < \infty$$

where

$$\|f_p\| \equiv \left(\int_\Omega |f|^p dx \right)^{\frac{1}{p}}$$

Holder's inequality for

$$\frac{1}{p} + \frac{1}{q} = 1$$

is

$$|\langle f, g\rangle| \leq \|f\|_p \, \|g\|_q$$

with standard L^2 inner product.

when $p = q = 2$

$$|\langle f, g\rangle| \leq \|f\|_2 \, \|g\|_2$$

Longhand one can write (squaring above)

$$\left| \int_\Omega f(x) g(x) dx \right|^2 \leq \|f\|^2 \, \|g\|^2$$

<u>EXAMPLE</u> $D(L) = BC[0,\infty)$ <u>bnd</u> continuous on $[0,\infty)$

$$Lu \equiv \frac{1}{t}\int_0^t u(s)\,ds \qquad \text{(average)}$$

L'Hopital's rule implies $u(0^+)$ defined!
Bounded under <u>sup norm</u> $\|u(t)\| = \sup_{t \geq 0} |u(t)|$

$$\left|\frac{1}{t}\int_0^t u(s)\,ds\right| \leq \frac{1}{t}\int_0^t |u(s)|\,ds \leq \|u\|$$

Hence L bounded on a Banach Space

<u>EXAMPLE</u> Derivative operator $H = L^2[0,2\pi]$

$$D(L) = C^1[0,2\pi]$$

$$Lu \equiv \frac{du}{dx} \qquad \text{unbounded}$$

Easy to show using $u_n = \sin(nx)$, $u_n' = n\cos(nx)$

$$\|u_n\|^2 = \int_0^{2\pi} \sin^2 nx\,dx = \pi$$

$$\|Lu_n\|^2 = \int_0^{2\pi} n^2\cos^2 nx\,dx = n^2\pi$$

Hence

$$\frac{\|Lu_n\|^2}{\|u_n\|^2} = n^2 \to \infty$$

and $\nexists\, k$ s.t

$$\|Lu_n\| \leq k\|u_n\| \qquad \forall n.$$

Operators with nonclosed ranges $R(L)$

If $R(L)$ is closed then $H = R(L) \oplus R(L)^{\perp}$ needed for Fredholm alternative. If $R(L)$ is not closed decomposition may not be possible. For $H = \mathbb{R}^n$, $Lu = Au$, $A \in \mathbb{R}^{n\times n}$ the range is closed

EXAMPLE Even some bounded L have $R(L) \neq \overline{R(L)}$

$$Lu \equiv \sum_{n=1}^{\infty} \frac{1}{n} \langle u, \phi_n \rangle \phi_n$$

where $\{\phi_n\}$ orthonormal basis. Parseval equality $\Rightarrow$

$$\|Lu\|^2 = \sum_{n\geq 1} \frac{1}{n^2} |\langle u, \phi_n\rangle|^2 \leq \sum_{n\geq 1} |\langle u, \phi_n\rangle|^2 = \|u\|^2$$

shows L bounded on H.

$$u_n \equiv \sum_{K=1}^{n} \phi_K \qquad Lu_n = \sum_{K=1}^{n} \frac{1}{K} \phi_K \ , \quad \langle \phi_i, \phi_j \rangle = \delta_{ij}.$$

Clearly $Lu_n \to y \equiv \sum_{K=1}^{\infty} \frac{1}{K} \phi_K$. In particular $\|y\|^2 = \frac{\pi^2}{6}$.

However, $\nexists\, z \in H$ such that $Lz = y$. If there were

$$Lz = \sum_{n=1}^{\infty} \frac{1}{n} \langle z, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \frac{1}{n} \phi_n$$

or $\langle z, \phi_n \rangle = 1$ for all n. But then

$$z = \sum_{n=1}^{\infty} \phi_n \quad \text{diverges}$$

$$\boxed{\begin{array}{c} L \text{ bounded} \\ Lu_n \to y \in H \\ \nexists\, z \text{ s.t. } Lz = y \end{array}}$$

Operator Domains

whether L is bounded or unbounded the domain $D(L)$ must be a subspace of H.

If L is bounded it is almost always the case $D(L) = H$. If not its domain of definition can be extended.

Most unbounded L have 'dense' domains:

L bounded	$D(L) = H$
L unbounded	$D(L)$ dense in H

<u>Ex</u> $\quad Lu = Au \qquad D(L) = \mathbb{R}^n$

<u>Ex</u> $\quad Lu = \int_a^b K(x,y)u(y)\,dy \qquad D(L) = L^2[a,b]$ **

<u>Ex</u> $\quad Lu = u''$ may have a domain

$$D(L) = \{u \in C^2[a,b] : u(a) = u(b) = 0\}$$

which is dense in $L^2[a,b]$.

This domain is associated with BVP

$$u'' = f \qquad u(a) = u(b) = 0$$

** assuming $K \in H \times H$, $H = L^2[a,b]$.

Adjoint Operators: Theory

Let $L : D(L) \to H$. We say $L^* : D(L^*) \to H$ is the adjoint of L if

$$\langle Lu, v \rangle = \langle u, L^* v \rangle \qquad \forall u \in D(L), \forall v \in D(L^*)$$

For $H = \mathbb{R}^n$ the adjoint L^* of $Lu \equiv Au$ is easily defined for

$$L^* u = A^T u$$

Moreover $D(L) = D(L^*) = H$.

EXISTENCE FOR BOUNDED OPERATORS

If L is bounded operator on H then $\ell : H \to \mathbb{C}$, $\ell(u) \equiv \langle Lu, v \rangle$ is a bounded functional for each fixed.

From theory on functionals (bounded) the Riesz representation theorem implies for each $v \in H$, $\exists g$ s.t.

$$\langle Lu, v \rangle = \langle u, g \rangle \qquad \forall u \in H.$$

Since g depends uniquely on v we let

$$L^* v = g$$

namely an adjoint operator exists.

EXISTENCE FOR UNBOUNDED OPERATORS

Let $L : D(L) \to H$ be unbounded with a domain $D(L)$ dense in H.

Seek to find g such that

$$(1) \quad \langle Lu, v \rangle = \langle u, g \rangle, \quad \forall u \in D(L)$$

Clearly $(v, g) = (0, 0)$ works but seek g's for $v \neq 0$.

May be (1) is true for g_1, g_2. Badnews.

But, were this the case

$$(3) \quad \langle u, g_1 - g_2 \rangle = 0 \qquad \forall u \in D(L)$$

The denseness of $D(L)$ in H would then imply

$$g_1 = g_2$$

Since such g are unique we may define

$$L^* v = g \qquad v \in D(L^*)$$

Other than knowing $0 \in L^*$ we don't know the size of the domain $D(L^*)$

Self Adjoint and Symmetric Operators

Authors differ on how to define symmetric and self adjoint. Here I present that of Reed and Simon (I Functional Analysis)

<u>Definition</u>: L densely defined on H is <u>symmetric</u> if

(i) $D(L) \subset D(L^*)$

(ii) $Lu = L^* u \ , \quad \forall u \in D(L)$

Most applied mathematicians say L is <u>formally self adjoint</u> if (ii) holds on any subset of H (dense)

<u>Defn</u>: L densely defined on H is <u>selfadjoint</u> if

(i) $D(L) = D(L^*)$

(ii) $Lu = L^* u \qquad \forall u \in D(L)$

Selfadjoint is a big huge deal.

Symmetric operators are usually extendable to self adjoint ones and is technical detail in functional analysis.

EXAMPLE Periodic Boundary conditions

$$Lu \equiv u''$$

$$D(L) \equiv \{ u \in C^2[0,\pi] : u(0) = u(\pi),\ u'(0) = u'(\pi) \}$$

Again we integrate by parts

$$\langle Lu, v \rangle = \int_0^\pi u'' v\, dx$$

$$= u'v \Big|_0^\pi - \int_0^\pi u'v'\, dx$$

$$= u'v \Big|_0^\pi - uv' \Big|_0^\pi + \int_0^\pi u \underbrace{v''}_{L^*v} dx$$

Owing to periodicity of u, u' at $x = 0, \pi$

$$= \underbrace{u'(0)\,(v(\pi) - v(0)) - u(0)\,(v'(\pi) - v'(0))}_{\text{must vanish since } u'(0) \text{ and } u(0) \text{ are arb.}} + \langle u, L^* v \rangle$$

Yields to the adjoint operator

$$L^* v = v''$$

$$D(L^*) = \{ v \in C^2[a,b] : v(0) = v(\pi),\ v'(0) = v'(\pi) \}$$

Since $Lu = L^*u \ \forall u \in D(L^*) = D(L)$, <u>L is self adjoint</u>

Side note, $u(x) = \text{const}$ is in $D(L)$. But $u(x) = k \Rightarrow Lu = 0$ hence $N(L)$ nonempty and $Lu = f$ can't have unique soln

EXAMPLE $H = L^2$

$$Lu \equiv u'' + u'$$

$$D(L) \equiv \{ u \in C^2[0,1] : \; u(0) = 0, \; u'(1) = 0 \}$$

To find L^* we must find an L^* such that

$$\langle Lu, v \rangle = \langle u, L^* v \rangle$$

for an appropriate domain $D(L^*)$

$$\langle Lu, v \rangle = \int_0^1 (u'' + u) v \, dx$$

$$\langle Lu, v \rangle = \underbrace{(u' + u) v \Big|_0^1 - u v' \Big|_0^1}_{\text{need this to vanish } \forall u \in D(L)} + \int_0^1 \underbrace{(v'' - v')}_{L^* v} u \, dx$$

Since $u(0) = 0$, $u'(1) = 0$, algebra yields

$$\langle Lu, v \rangle = u'(0) \underbrace{v(0)} + \underbrace{(v(1) - v'(1))} \cdot u(1) + \int_0^1 (v'' - v') u \, dx$$

Domain $D(L^*)$ must have indicated terms vanish. Thus

$$\langle Lu, v \rangle = \langle u, L^* v \rangle$$

for

$$L^* v \equiv v'' - v'$$

$$D(L^*) = \{ u \in C^2[0,1] : \; v(0) = 0, \; v(1) - v'(1) = 0 \}$$

Note $L^* \neq L$ and $D(L^*) \neq D(L)$

EXAMPLE Integral operator (Hilbert Schmidt)

$$Lu \equiv \int_0^1 \sin(x+2y)\, u(y)\, dy$$

is defined on $D(L) = H = L^2[0,1]$. Since kernel $k(x,y) \equiv \sin(x+2y)$ is smooth on compact domain

$$\langle Lu, v\rangle = \int_0^1 v(x) \int_0^1 \sin(x+2y)\, u(y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 \sin(x+2y)\, v(x)\, u(y)\, dy\, dx$$

$$= \int_0^1 u(y) \underbrace{\int_0^1 \sin(x+2y)\, v(x)\, dx}_{L^* v}\, dy$$

$$= \langle u, L^* v\rangle$$

if

$$L^* v \equiv \int_0^1 \sin(x+2y)\, v(x)\, dx$$

Note $D(L^*) = H$ and

$$L^* u \neq Lu$$

not self adjoint.

EXAMPLE Laplacian on $H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^3$ compact

$$Lu \equiv \nabla^2 u$$

$$D(L) \equiv \{ u \in C^2(\Omega) : u|_{\partial\Omega} = 0 \}$$

By applying Divergence Theorem to $\vec{F} = v\nabla u - u\nabla v$ we get Green's identity (second):

$$\int_\Omega (v\nabla^2 u - u\nabla^2 v)\,dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) dS$$

Written another way

$$\langle Lu, v\rangle = \underbrace{\int_{\partial\Omega} v\frac{\partial u}{\partial n}\,ds}_{\substack{\text{must vanish} \\ \forall u \in D(L)}} - \underbrace{\cancelto{0}{\int_{\partial\Omega} u\frac{\partial v}{\partial n}\,ds}}_{u \in D(L)} + \int_\Omega u \underbrace{\nabla^2 v}_{L^* v}\,dx$$

Conclude $\langle Lu, v\rangle = \langle u, Lv\rangle \quad \forall u \in D(L), v \in D(L^*)$ if

$$L^* v = \nabla^2 v$$

$$D(L^*) = D(L)$$

hence L is self adjoint

EXAMPLE **Volterra integral operator**

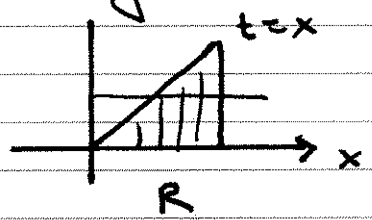
$$Lu \equiv \int_0^x u(t)\,dt \qquad D(L) = L^2[0,1]$$

Then

$$\langle Lu, v\rangle = \int_0^1 \left(\int_0^x u(t)\,dt\right) v(x)\,dx$$

$$= \int_0^1\int_0^x u(t)v(x)\,dt\,dx = \iint_R u(t)v(x)\,dx\,dt$$

Integration region:



$$t < x < 1$$
$$0 < t < 1$$

Thus

$$\langle Lu, v\rangle = \int_0^1 u(t)\left(\int_t^1 v(x)\,dx\right) dt$$

and

$$L^* v = \int_x^1 v(t)\,dt$$

Clearly $L^* \neq L$

EXAMPLE Let $H_1 = L^2[0,1]$ with its standard inner product $\langle \cdot, \cdot \rangle$. If we let $H = H_1 \otimes H_1$ the inner product for this Hilbert space is

$$\langle\langle u, v \rangle\rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle$$

where $u = (u_1, u_2)$ etc.

Consider $L : H \to H$ defined by

$$Lu = \frac{d^2 u}{dx^2} + Au \qquad A \in \mathbb{R}^{2\times 2}$$

$$D(L) = \{ u \in H : Lu \text{ defined}, u_k(0) = u_k(1) = 0 \}$$

Find adjoint

$$\langle Lu, v \rangle = \langle u_1'' + a_{11}u_1 + a_{12}u_2, v_1 \rangle + \langle u_2'' + a_{21}u_1 + a_{22}u_2, V_2 \rangle$$

With some foresight $D(L^*) = D(L)$ so integrating by parts

$$\langle Lu, v \rangle = \langle u_1, v_1'' \rangle + \underline{\langle a_{11}u_1 + a_{12}u_2, v_1 \rangle} + \langle u_2, v_2'' \rangle + \underline{\langle a_{21}u_1 + a_{22}u_2, v_2 \rangle}$$

underlined part from $\langle\langle Au, v \rangle\rangle$. Claim this equals $\langle\langle u, A^T v \rangle\rangle$ so that

$$\langle Lu, v \rangle = \langle \frac{d^2 v}{dx^2} + A^T v, u \rangle$$

implies

$$L^* v = \frac{d^2 v}{dx^2} + A^T v$$

Operators: definitions and properties

In the following definitions and theorems, H is a Hilbert space though many definitions hold for normed linear spaces.

Sets: Let $S \subset H$ be some set. Then

(C1) S bounded	$\Leftrightarrow$	$\exists M > 0$ s.t. $\parallel x \parallel \leq M, \forall x \in S$
(C2) S compact	$\Leftrightarrow$	Every sequence $\{x_n\} \subset S$ contains a convergent subsequence $\{x_{n_k}\}$ which converges to $x \in S$
(C3) S bounded	$\not\Rightarrow$	S compact
(C4) S (sequentially) compact	$\Rightarrow$	S closed and bounded
(C5) $S \equiv \{x \in H : \parallel x \parallel \leq 1\}$ compact	$\Rightarrow$	$dim(H) < \infty$

Definition: Bounded Operator An operator $L : H \to H$ is bounded if there exists some $M > 0$ such that

$$\parallel Lx \parallel \leq M \parallel x \parallel \quad \forall x \in H \tag{1}$$

If L is not bounded then L is unbounded.

Definition: Compact Operator An operator $L : H \to H$ is compact if it maps bounded sets into compact sets. Equivalently, for every bounded $\{x_n\} \subset H$ the sequence $\{Lx_n\}$ has a convergent subsequence.

Definition: Adjoint Operator Let $L : D(L) \subset H \to H$. The operator L^* defined on $D(L^*)$ is the adjoint of L if

$$< Lu, v > = < u, L^* v > \quad , \quad \forall u \in D(L), \forall v \in D(L^*)$$

Further we say L is <u>self adjoint</u> if

$$\begin{aligned} Lu &= L^* u \quad , \quad \forall u \in D(L) \\ D(L) &= D(L^*) \end{aligned}$$

L is <u>formally self adjoint</u> if $Lu = L^* u$ on some set dense subset of H.

The Reisz representaion Theorem assures that bounded operators always have adjoints. Since compact operators are bounded the same applies to them.

Theorems for Compact and Bounded linear operators

(1)	L bounded	$\Leftrightarrow$	L continuous
(2)	L bounded	$\Leftrightarrow$	$L\,(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} Lu_n$
(3)	L bounded and $R(L)$ closed	$\Leftrightarrow$	$R(L) \perp N(L^*)$
(4)	L compact	$\Rightarrow$	L bounded
(5)	L linear, $dim(R(K)) < \infty$	$\Rightarrow$	L compact
(6)	L bounded, $\{\phi_n\}_{n=1}^{\infty}$ orthonormal	$\Rightarrow$	$\lim_{N\to\infty} L\phi_n = 0$
(7)	L_1, L_2 bounded	$\Rightarrow$	$L_1 + L_2$ bounded
(8)	L_1, L_2 compact	$\Rightarrow$	$L_1 + L_2$ compact
(9)	L_n compact, $\parallel L_n - L \parallel_{op} \to 0$	$\Rightarrow$	L compact

Definition: Operator norm For any bounded operator $L : H \to H$ we define the operator norm as

$$\parallel L \parallel_{op} \equiv \max_{\parallel x \parallel = 1} \parallel Lx \parallel = \max_{x \neq 0} \frac{\parallel Lx \parallel}{\parallel x \parallel} \tag{2}$$

Theorems above imply that the space of bounded (and compact) operators are normed linear spaces. Completeness can also be shown.

Self Adjoint Compact Operators

The classic example of a self adjoint compact operator is the Hilbert-Schmidt operator L defined on $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is compact.

$$Lu = \int_\Omega k(x,y)u(y)\, dy$$

This is well defined if kernel k is square integrable on Ω^2 and is self adjoint if

$$k(x,y) = k(y,x)$$

Below L is a self adjoint compact operator on H with eigenvalue-vector pairs:

$$L\phi_n = \lambda_n \phi_n \qquad , \qquad \phi_n \neq 0$$

a) All the eigenvalues of L are real

b) If $\lambda \neq 0$ then the associated eigenspace has $dim E_\lambda < \infty$

c) If $\lambda_1 \neq \lambda_2$ then $< u_1, u_2 >= 0$.

d) Either L has a finite number of eigenvalues or a countable number with $\lambda_n \to 0$.

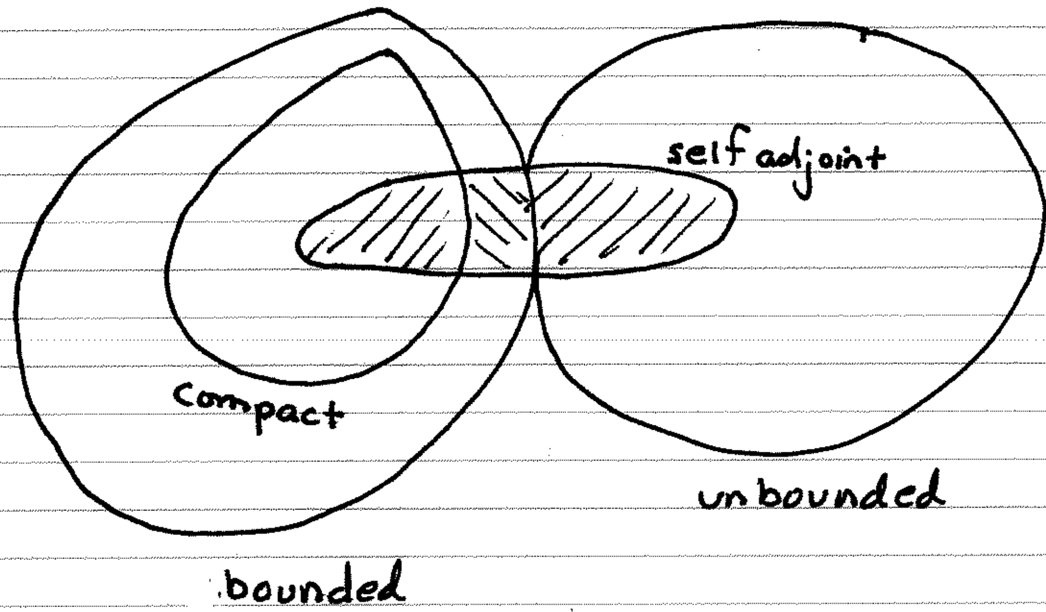
e) $\exists$ a sequence $\{\phi_n\}$ of orthonormal eigenfunctions of L that are a basis for H [1]

Note that $\lambda = 0$ is very possible hence $N(L)$ may be nonempty. In the case of "degenerate" integral operators one can even have $dimN(L) = \infty$. Statement c) above is not true for finite dimensional Hilbert spaces. It applies if $dim(H) = \infty$ regardless of whether $N(L)$ is empty or not. The last fact e) is very important. In practice one uses Gram Schmidt to orthogonalize bases for each eigenspace $E_\lambda(L)$. Then, if L is a self adjoint compact operator then the eigenvalues form an orthonormal basis for H and one can always write:

$$f = \sum_{n=1}^{\infty} < f, \phi_n > \phi_n \qquad , \qquad \forall f \in H$$

[1] Reed and Simon, Functional Analysis I, pg 203

Operator Space nestings



A separate issue of importance in Functional analysis is whether:

R(L) closed

R(L) notclosed.

Fredholm Alternative (Bounded L, closed $R(L)$)

Let $L: H \to H$ be bounded and $R(L)$ closed.

$$Lu = f \text{ has soln} \iff \langle f, v\rangle = 0 \quad \forall v \in N(L^*)$$

Proof. First let $Lu = f$. Then

$$\langle Lu, v\rangle = \langle u, L^* v\rangle = \langle f, v\rangle = 0 \qquad \forall v \in N(L^*)$$

For the opposite implication we note that since $R(L)$ is closed

"closed"

$$H = R(L) \oplus R(L)^\perp$$

Then $f = f_r + f_r^\perp$ where $f_r \in R(L)$.

"bounded"

$$\langle f_r^\perp, Lz\rangle = 0 \qquad \forall z \in H$$

$$\langle L^* f_r^\perp, z\rangle = 0 \qquad \forall z \in H$$

so that $f_r^\perp \in N(L^*)$

$$\langle v, f\rangle = 0 \qquad \forall v \in N(L^*)$$

$$\langle f_r^\perp, f_r^\perp + f_r\rangle = 0$$

$$\| f_r^\perp \|^2 = 0$$

$$f_r^\perp = 0$$

Conclude $f = f_r + \cancel{f_r^\perp} = f_r$ hence $f \in R(L)$ /

EXAMPLE Integral equation

$$(1) \qquad u(x) = f(x) + \lambda \int_0^1 xy\, u(y)\, dy$$

Define the (compact, bounded) operator

$$\underline{K} u \equiv \int_0^1 xy\, u(y)\, dy$$

the (1) is

$$(2) \qquad (\underline{I} - \lambda \underline{K}) u = f$$

The operator on the left is bounded (and closed range... technical) so solvability of (1) depends on (F.A.Th.) $N(\underline{I}-\lambda\underline{K})^*$

Clearly

$$(\underline{I} - \lambda \underline{K})^* = (\underline{I} - \lambda \underline{K})$$

Suffice to find basis for $N(\underline{I} - \lambda \underline{I})$ $\Longleftrightarrow$

$$u(x) = \lambda x \int_0^1 u(y)\, dy = a x \qquad a \in \mathbb{R}$$

i.e. $N(\underline{I} - \lambda \underline{I}) = \text{span}\{x\}$.

Fred. Alt. Theom implies (1) has a soln $\Leftrightarrow$

$$\int_0^1 x f(x)\, dx = 0$$

EXAMPLE Integral Eqn perturbed eigenvalues

$$\mathbb{K}u \equiv (\mathbb{K}_0 + \varepsilon \mathbb{K}_1) u$$

$$\mathbb{K}u \equiv \int_0^1 xy\,u(y)\,dy + \varepsilon \int_0^1 u(y)\,dy$$

Unperturbed eigenvalue problem

$$(\mathbb{K}_0 - \lambda_0 I) u_0 = 0$$

where

$$\lambda_0 u_0 = \int_0^1 xy\,u(y)\,dy = ax \qquad a \in \mathbb{R}.$$

Hence $u_0(x)$ proportional to x. wlog $u_0(x) = x$

$$\lambda_0 = \int_0^1 y^2\,dy = \tfrac{1}{3}$$

Conclude $\lambda_0 = \frac{1}{3}$ is the (nonzero) e-value

$$\lambda_0 = \tfrac{1}{3} \qquad N(\mathbb{K}_0 - \lambda_0 I) = \mathrm{span}\{x\}$$

Perturbed eigenvalue problem.

Assume the expansions

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \cdots$$

$$u = u_0 + \varepsilon u_1 + \cdots.$$

and use F.A.Th. to find λ_1.

Collect like powers of ε in

$$(\mathbb{K}_0 + \varepsilon \mathbb{K}_1)(u_0 + \varepsilon u_1 + \cdots) = (\lambda_0 + \varepsilon\lambda_1 + \cdot)(u_0 + \varepsilon u_1 + \cdots)$$

yields

$$(\mathbb{K}_0 - \lambda_0 I)\, u_0 = 0$$

(1) $$(\mathbb{K}_0 - \lambda_0 I)\, u_1 = \lambda_1 u_0 - \mathbb{K}_1 u_0 \equiv f$$

Apply F.A.T. to (1). Note $(K_0 - \lambda I)^* = (K_0 - \lambda I)$ so

$$\langle \lambda_1 u_0 - \mathbb{K}_1 u_0 , v \rangle = 0 \qquad \forall v \in N(K_0 - \lambda I)$$

Previously found $N(\mathbb{K}_0 - \lambda I) = \text{span}\{x\}$

$$\langle \lambda_1 x - \mathbb{K}_1(x) , x \rangle = 0$$

$$\langle \lambda_1 x - \tfrac{1}{2} , x \rangle = 0$$

$$\lambda_1 \|x\|^2 - \tfrac{1}{2}\langle 1, x \rangle = 0$$

$$\tfrac{1}{3}\lambda_1 - \tfrac{1}{2}\cdot\tfrac{1}{2} = 0$$

$$\boxed{\lambda_1 = \tfrac{3}{4}}$$

Fredholm Alternative ($D(L)$ dense in H)

$$Lu = f \text{ has soln } \Rightarrow \langle f, v \rangle = 0 \quad \forall v \in N(L^*)$$

Pf: Trivial. Denseness implies L^* is uniquely defined on some $D(L^*)$ though it does not state how "big" $D(L^*)$ is. Then

$$\langle Lu, v \rangle = \langle u, L^* v \rangle = \langle f, v \rangle \qquad \square$$

Indicated terms vanish $\forall v \in D(L^*)$

Remark: In a compact form

$$f \in R(L) \Rightarrow f \in N(L^*)^{\perp}$$

Non equivalence does not hinder the usefulness as "necessary" conditions for solns may still be derived.

EX **Helmholtz on rectangle $\Omega = [0,\pi]^2$**

$$Lu \equiv \nabla^2 u + 5u$$

$$D(L) = \{u \in C^2(\Omega) : u|_{\partial\Omega} = 0\}$$

L is "a" Helmholtz operator. Readily verified with Green's identity that L is self adjoint.

$$(1) \qquad \nabla^2 u + 5u = f \qquad u \in D(L)$$

where $f = \sin(2x)$. Question is, does (1) have a solution? Has a soln only if $f \in N(L^*)^\perp$.

One can verify $T \equiv \text{span}\{\overset{v_1}{\sin 2x \sin y}, \overset{v_2}{\sin x \sin 2y}\} \subset N(L^*)$
Hard to prove but $T = N(L) = N(L^*)$.

Fred. Alt Theorem implies f must be orthogonal to both v_1 and v_2.
Calculations show

$$\langle v_1, f \rangle = \pi \neq 0$$

$$\langle v_2, f \rangle = 0$$

hence (1) has no solution if $f = \sin(2x)$.
Recall too in the above

$$\langle u, v \rangle = \int_0^{\pi}\int_0^{\pi} u(x,y)\, v(x,y)\, dy\, dx$$

EX **Helmhotz on rectangle** $\Omega \equiv [0,\pi]^2$

$$Lu \equiv \nabla^2 u + u$$

where the domain for L is

$$D(L) = \{u \in C^2(\Omega) : u|_{\partial\Omega} = 0\}$$

L is a Helmholtz operator. It's not hard to show that L is self adjoint.

Consider the problem

not in $D(L)$

$$(1) \quad \nabla^2 u + u = f \qquad u|_{\partial\Omega} = 0$$

where $f(x,y) = \cos(2x)$.

Eqn (1) has a solution only if $f \in N(L^*)^\perp$ by F.A. Thm.

One can verify $T \equiv \text{span}\{\sin x, \sin y\} \subset N(L^*)$. Actual equality by harded to prove.

$$\left.\begin{aligned} \langle \sin x, f\rangle &= -\frac{2\pi}{3} \\ \langle \sin y, f\rangle &= 0 \end{aligned}\right\} \text{ not both zero}$$

Hence (1) does not have a soln if $f = \cos(2x)$

$$\langle u, v\rangle \equiv \int_0^\pi\!\!\int_0^\pi u(x,y)\,v(x,y)\,dy\,dx$$

Resolvents $(\mathbb{I} - \lambda \mathbb{K})^{-1}$

Let $\mathbb{K}$ be compact and self adjoint. Seek a series solution to

(1) $$(\mathbb{I} - \lambda \mathbb{K})\, u = f$$

Such a solution is unique $\Leftrightarrow N(\mathbb{I} - \lambda \mathbb{K}) = 0$ in which case one might write

(2) $$u = (\mathbb{I} - \lambda \mathbb{K})^{-1} f$$

The operator on the right is called the resolvent operator

An example of such an equation would be the following integral eqn

$$u(x) - \lambda \int_a^b k(x,y)\, u(y)\, dy = f(x)$$

where the Hilbert Schmidt kernel is symmetric

$$k(x,y) = k(y,x)$$

This assures the operator

$$\mathbb{K} u \equiv \int_a^b k(x,y)\, u(y)\, dy$$

is self adjoint and has a complete set of orthonormal eigen functions

(2) $$\mathbb{K} \phi_n = \mu_n \phi_n \qquad \langle \phi_i, \phi_j \rangle = \delta_{ij}$$

To solve we let

(3) $$u = f + g$$

in (1) and rearrange to find

(4) $$(\mathbb{I} - \lambda \mathbb{K}) g = \lambda \mathbb{K} f$$

On account of $\{\phi_n\}$ being complete ($\mathbb{K}$ comp, s. adj)

$$f = \sum f_n \phi_n \qquad f_n = \langle f, \phi_n \rangle$$

$$g = \sum g_n \phi_n \qquad g_n = \langle g, \phi_n \rangle$$

So that (4) implies

(5) $$\sum_{n=1}^{\infty} (1 - \lambda \mu_n) g_n \phi_n = \sum_{n=1}^{\infty} \lambda \mu_n f_n \phi_n$$

orthogonality of $\phi_n \Rightarrow$

(6) $$(1 - \lambda \mu_n) g_n = \lambda \mu_n f_n \qquad \forall n$$

Considering $u = f + g$ and (6) yields g_n

(7) $$u(x) = f(x) + \lambda \sum_{n \geq 1} \frac{\mu_n}{1 - \lambda \mu_n} \langle f, \phi_n \rangle \phi_n = (\mathbb{I} - \lambda \mathbb{K})^{-1} f$$

$$u(x) = f(x) + \lambda \int_a^b r(x, y, \lambda) f(y) \, dy$$

The last written expression only if $\mathbb{K}$ is an integral operator. (7) defines resolvent.