Hilbert Spaces

Defn: A Hilbert space $H$ is an inner product which is complete w.r.t.

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

Below we give several examples of Hilbert Spaces

**EXAMPLE**

$$H = L^2(\mathbb{R}) \quad \langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} \, dx$$

with the understanding $f = g$

is in the a.e. Lebesgue sense

**EXAMPLE**

$$H = L^2(\mathbb{R}) \quad \langle f, g \rangle = \int_{\mathbb{R}} r f(x) \overline{g(x)} \, dx$$

Providing $r$ is integrable and

$$0 < r(x) \leq M \quad \forall x \in \mathbb{R}$$

**EXAMPLE**

$$H = l^2 = \{ \{x_n\} : x_n \in \mathbb{C}, \sum_{n=1}^{\infty} |x_n|^2 < \infty \}$$

Here the inner product is

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

where $x, y$ are sequences.

All above stated without proof of completeness.
**EXAMPLE (Sobolev Spaces)**

\[
\langle f, g \rangle = \int_a^b \sum_{j=0}^n f^{(j)}(x) g^{(j)}(x) \, dx
\]

is an inner product sub space of \( L^2[a,b] \). Following is a Hilbert space

\[
H^n[a,b] = \left\{ f \in L^2[a,b] : \|f\| < \infty \right\}
\]

Note that \( H^n[a,b] \) is a proper subset of \( L^2[a,b] \) for \( n > 1 \).

**EXAMPLE** Let \( C^0_0(\mathbb{R}) \) be set of all \( C^0 \) functions of compact support.

The completion of \( C^0_0(\mathbb{R}) \) in \( H^n[a,b] \) is defined as \( H^n(\mathbb{R}) \) and is a Hilbert space.
EXAMPLE Non complete inner product space

\[ X = \{ f \in C[0,1] : \|f\| < \infty \} \]

using standard \( L^2 \) inner product

\[ f_n(x) \]

Explicit formula for \( \{f_n\} \) sequence

\[
f_n(x) = \begin{cases} 
0 & x \in (0, \frac{1}{2} - \frac{1}{n}) \\
\frac{1}{2} + \frac{n}{2} (x - \frac{1}{2}) & x \in (\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}) \\
1 & x \in (\frac{1}{2} + \frac{1}{n}, 1) 
\end{cases}
\]

Very clear that

\[ f_n(x) \to H(x - \frac{1}{2}) \in X \]

i.e., a Cauchy sequence that does not converge to an element of \( X \)

Remark Recall convergence implies Cauchy. \( \exists \) \( N \) s.t.

\[ \|f_n - f_m\| \leq \|f_n - f\| + \|f_m - f\| \leq 2\varepsilon \]

\[ \forall n, m \geq N \text{ since } \]

\[ \|f_n - f\| \leq \varepsilon \quad \forall n \geq N \]

if \( f_n \to f \).
**Banach versus Hilbert spaces**

A Banach space is a complete normed space.

\[ X = \{ f \in C[a,b] : \| f \| < \infty, \| f \| = \max_{a \leq x \leq b} | f(x) | \} \]

is a linear space with a norm not induced by a metric. It is also complete (below) hence is a Banach space.

However, \( X' = C[a,b] \) with the \( L^2 \) inner product is not a Hilbert space. There are many continuous \( \{ f_n \} \) where \( f_n \to f \) in \( X' \). Hence, the unidirectional implication

**Hilbert \( \Rightarrow \) Banach**

**Completeness proof**

Let \( \{ f_n \} \subset X \) be Cauchy in \( X \). Then for each \( x \in [a,b] \), \( \{ f_n(x) \} \) is a Cauchy sequence in \( \mathbb{R} \). Since \( \mathbb{R} \) complete, \( \exists f(x) \) s.t.

\[ f_n(x) \to f(x) \quad \text{pointwise} \]

To show \( f(x) \in X \), i.e. continuous:

\[ |f(t) - f(t+s)| \leq |f(t) - f_k(t)| + |f_k(t) - f_k(t+s)| + |f_k(t+s) - f(t+s)| \]

\[ \downarrow \quad \text{cont. of } f_k(t) \quad \downarrow \]

0
Approximation by finite sets (same as before)

Let \( H \) be a real Hilbert space with \( \langle x, y \rangle = \langle y, x \rangle \)

Let \( T = \{ \varphi_k \}_{k=1}^{n} \subset H \) and \( f \in H \). Here the functions \( \varphi_k \) need not be orthogonal.

\[
F(\alpha) = \| f - \sum_{k=1}^{n} \alpha_k \varphi_k \|^2
\]

As before \( \frac{\partial F}{\partial \alpha_k} = 0 \) yields a system for \( \alpha_k \):

\[
(2) \quad \Phi \alpha = \beta
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n)^T \) and

\[
[\Phi]_{ij} = \langle \varphi_i, \varphi_j \rangle \quad \beta_j = \langle f, \varphi_j \rangle
\]

If \( \varphi_k \) are independent, eqn (2) has a unique soln. Defn of ind.

\[
\sum_{i=1}^{n} \alpha_i \varphi_i = 0 \quad \Leftrightarrow \quad \alpha_i = 0 \quad \forall i
\]

\[
\sum_{i=1}^{n} \alpha_i \langle \varphi_i, \varphi_j \rangle = 0 \quad \Leftrightarrow \quad \alpha_i = 0 \quad \forall i
\]

\[
\Phi \alpha = 0 \quad \Leftrightarrow \quad \alpha = 0
\]

The solution of (2) yields the least squares approximation relative to the independent set \( T \). The approximator is in \( \text{span}(T) \).
**Example** \[ H = L^2(-1, 1) \text{ and } f(x) \equiv 2H(x) - 1 \]

where \( H(x) \) is the Heaviside fn.

\[ \varphi_i(x) \equiv x^{2i-1}, \quad i = 1, 2, 3 \]

are odd monomials. Least squares approximation

\[ \overline{f}(x) = \sum_{i=1}^{n} a_i \varphi_i(x) \]

where \( \mathbf{a} \) a solution of \( \overline{A} \mathbf{a} = \mathbf{b} \)

(self adjoint) \( \overline{A}_{ij} = \langle \varphi_i, \varphi_j \rangle \quad \mathbf{b}_i = \langle \varphi_i, f \rangle \)

For \( n = 1, 2, 3 \) can compute:

\[ n=1 \quad \overline{A} = \begin{bmatrix} \frac{3}{2} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \end{bmatrix} \]

\[ n=2 \quad \overline{A} = \begin{bmatrix} \frac{2}{3} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{7} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1, \frac{1}{2} \end{bmatrix}^T \]

\[ n=3 \quad \overline{A} = \begin{bmatrix} \frac{2}{3} & \frac{2}{5} & \frac{2}{7} \\ \frac{2}{5} & \frac{2}{7} & \frac{2}{9} \\ \frac{2}{7} & \frac{2}{9} & \frac{2}{11} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1, \frac{1}{2}, \frac{1}{3} \end{bmatrix}^T \]

For example solving \( \overline{A} \mathbf{a} = \mathbf{b} \) for \( n = 1, 2 \) we find \( \mathbf{a} : \)

\[ n=1 \quad \overline{f}_1(x) = \frac{3}{2} x \]

\[ n=2 \quad \overline{f}_2(x) = \frac{5}{16} (9x - 7x^3) \]

General notes

\[ \langle \varphi_i, \varphi_j \rangle = \frac{2}{2(i+j)-1} \]
Approximation by finite orthonormal sets

As before

\[ F(d) = \| f - \sum_{k=1}^{n} \alpha_k \phi_k \|_2 \]

is minimized by any \( \alpha \) that satisfies

\[ \Phi \alpha = \beta \quad \Phi_{ij} = \langle \phi_i, \phi_j \rangle \quad \beta_i = \langle \phi_i, f \rangle \]

If we now assume \( \{ \phi_i \} \) is an orthonormal set then

\[ \langle \phi_i, \phi_j \rangle = \delta_{ij} \quad \text{(Kronecker delta)} \]

in which case \( \Phi = I \), the identity matrix.

\[ f(x) \approx \hat{f}_n(x) = \sum_{i=1}^{n} \langle \phi_i, f \rangle \beta_i \]

In this context

\[ \langle \phi_i, f \rangle = \text{Fourier coefficients} \]

\[ \hat{f}_n(x) = \text{Fourier series (generalized)} \]

**Key issue**: does the series converge to \( f(x) \)?
Approximation Error, and Bessel’s Inequality

Let \( \{ \phi_i \} \) be an orthonormal set on some Hilbert space \( H \).

\[
E(x) = \| f - \sum_{i=1}^{n} \delta_{j} \phi_{j} \|^{2}, \quad \delta_{j} = \langle \phi_{j}, f \rangle
\]

\[
E(x) = \langle f - \sum_{j} \delta_{j} \phi_{j}, f - \sum_{j} \delta_{j} \phi_{j} \rangle
\]

\[
E(x) = \| f \|^{2} - 2 \sum_{i} \delta_{i} \langle \phi_{i}, f \rangle + \sum_{i} \sum_{j} \delta_{i} \delta_{j} \langle \phi_{i}, \phi_{j} \rangle
\]

\[
E(x) = \| f \|^{2} - 2 \sum_{i} \delta_{i}^{2} + \sum_{i} \delta_{i}^{2}
\]

\[
E(x) = \| f \|^{2} - \sum_{i=1}^{n} \langle f, \phi_{i} \rangle^{2}
\]

Summarize

\[
E(x) = \| f \|^{2} - \sum_{i=1}^{n} \langle f, \phi_{i} \rangle^{2} \geq 0
\]

Given \( E(x) \geq 0 \) we then have Bessel’s Inequality

\[
\sum_{i=1}^{n} \langle f, \phi_{i} \rangle^{2} \leq \| f \|^{2}, \quad \forall n
\]

This series converges as \( n \to \infty \)
Fourier Series Convergence

Let \( \{ \phi_i \}_{i=1}^{\infty} \) be orthonormal set in \( H \)

Fourier series

\[
\overline{f}_n = \sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i \quad d_i = \langle f, \phi_i \rangle
\]

Claim \( \{ \overline{f}_n \} \) Cauchy in \( H \), wlog \( n > m \)

\[
\| \overline{f}_n - \overline{f}_m \|^2 = \left\| \sum_{i=1}^{n} d_i \phi_i - \sum_{j=1}^{m} d_j \phi_j \right\|^2 = \sum_{i=m+1}^{n} \| \phi_i \|^2 \leq n \sup_{i} \| \phi_i \|^2 = n
\]

\[
\leq \sum_{m+1}^{\infty} \langle f, \phi_i \rangle^2
\]

The infinite sum exists on account of Bessel's inequality. Moreover, it can be made small, i.e.

\[
\| \overline{f}_n - \overline{f}_m \|^2 \leq \sum_{m+1}^{\infty} \| f, \phi_i \|^2 \leq \epsilon, \quad \forall m, n > N
\]

Since \( \{ \overline{f}_n \} \) is Cauchy in a complete space \( H \), there exists an \( \overline{f} \) such that

\( \overline{f}_n \rightarrow \overline{f} \).
Incomplete orthonormal sets.

Not all countable orthonormal sets \( \{ \phi_n \} \) form a basis for \( H \).

\[ \text{span} \{ \phi_n \} \neq H \]

This is best seen by way of examples.

**Ex** \( H = L^2[0, 2\pi] \) and \( \{ \phi_n(x) \} = \{ \frac{1}{\sqrt{\pi}} \sin(nx) \} \)

One can verify \(0 < \frac{2\pi}{0} \)

\[ \langle \phi_n, \phi_m \rangle = \int_0^{2\pi} \sin(mx) \sin(nx) \, dx = 0 \quad m \neq n \]

\[ \langle \phi_n, \phi_n \rangle = \int_0^{2\pi} \frac{1}{\pi} \sin^2(nx) \, dx = 1 \]

But for \( f(x) = 1 \) we have

\[ \langle \phi_n, f \rangle = \int_0^{2\pi} 1 \cdot \sin(nx) \, dx = 0 \]

So that the Fourier series

\[ f_n(x) = \sum_{k=1}^{\infty} \langle \phi_k, f \rangle \phi_k = 0 \quad \forall n \]

Here \( f_n(x) \to 0 \neq f(x) \).
Example: Legendre polynomials \( \{ P_n(x) \} \)

Can be found by using Gram-Schmidt of monomials \( \{ x^n \} \) on \( L^2[-1,1] \).

Can be shown (not here) that \( \{ P_n(x) \} \)

is an orthogonal (not normalized) spanning set for \( L^2[-1,1] \). They are solutions to Legendre's differential equation

\[
\frac{d}{dx} \left[ (1-x^2) \frac{du}{dx} \right] + n(n+1) u = 0
\]

Such ODE's arise in physical problems involving spheres.

Given \( T = \{ P_n(x) \}_{n \geq 0} \) we define

\[
T' = \{ P_n(x) \}_{n = 2m+1}
\]

odd power polynomials in \( T \).

If we attempted a Fourier series of \( f(x) = 1 \) using \( T' \subset T \) we'd have

\[
\langle 1, P_n(x) \rangle = \int_{-1}^{1} P_n(x) dx = 0 \quad \text{for even } P_n(x) \text{ odd}
\]

So that

\[
f_n(x) = \sum_{m=0}^{\infty} \langle P_{2m+1}, f \rangle P_{2m+1} \equiv 0 \quad \forall n
\]

Again \( f_n(x) \to 0 \neq f(x) = 1 \).
Defn. An orthonormal set \( \{ \phi_n \}_{n=1}^{\infty} \) is complete if
\[
f = \sum_{n=1}^{\infty} <f, \phi_n> \phi_n \quad \forall f \in H.
\]

**Theorem.** The following are equivalent for orthonormal set \( \{ \phi_n \} \):

1. \( \{ \phi_n \}_{n=1}^{\infty} \) is a complete orthonormal set for \( H \)
2. \( f = \sum_{n=1}^{\infty} <f, \phi_n> \phi_n \quad \forall f \in H \)
3. \( \|f\|^2 = \sum_{n=1}^{\infty} |<f, \phi_n>|^2 \quad \forall f \in H \) (Parseval's equality)
4. \( <f, \phi_n> = 0 \quad \forall n \Rightarrow f = 0 \)

**Proof (2) \(\Rightarrow\) (3)** Due to Bessel's inequality \( \sum_{n=1}^{\infty} |<f, \phi_n>|^2 < \infty \),

\[
\|f\|^2 = \sum_{n=1}^{\infty} |<f, \phi_n>|^2 \sum_{m=1}^{\infty} |<f, \phi_m>|^2 \Rightarrow \sum_{n=1}^{\infty} |<f, \phi_n>|^2
\]

**Proof (3) \(\Rightarrow\) (4)** If \( <f, \phi_n> = 0 \quad \forall n \), Parseval's equality immediately gives

\( \|f\|^2 = 0 \quad \Rightarrow \quad f = 0 \).

See Naylor/Sell pg 307 (Linear Operator Theory in Engineering and Science) for complete proofs.
Operators on Hilbert Spaces

Given some Hilbert space any mapping

\[ L : H \rightarrow H \]

is an operator. In general the domain of \( L \) need not be equal to \( H \)

\[ D(L) \subseteq H \quad \text{domain} \]

\[ R(L) \subseteq H \quad \text{range} \]

we will also restrict our attention to linear operators

\[ L (\alpha u_1 + \beta u_2) = \alpha L(u_1) + \beta L(u_2) \]

for all \( \alpha, \beta \in \mathbb{C} \) and \( u_k \in D(L) \)

**EXAMPLE** \( H = \mathbb{R}^n \) (matrix multiply)

\[ Lu = Au \quad \text{A} \in \mathbb{R}^{n \times n} \]

Here \( D(L) = \mathbb{R}^n \).

**EXAMPLE** Integral operators

Let \( \Delta = [a, b] \) and define

\[ Ku = \int_{a}^{b} k(x, y)u(y)dy \]

for some continuous \( k(x, y) \) "kernel"
For example,

\[ K(u) = \int_0^1 \log(x+y) u(y) \, dy \]

For \( u(y) \equiv 1 \),

\[ K(u) = \int_0^1 \log(x+y) \, dy \]

can be computed using IBP

\[ K(u) = (x+1) \ln(x+1) - (x+1) \]

For \( x \in [0, 1] \) this is well defined and in \( H = L^2[0, 1] \).

**EXAMPLE** More general integral operators

\[ \Omega \subset \mathbb{R}^n \text{ compact} \]

\[ K(u) = \int_\Omega k(x, x') u(x') \, dx' \]

For appropriate \( k \), \( K : L^2(\Omega) \to L^2(\Omega) \)

Should be contrasted with a functional on a Hilbert Space such as

\[ F(u) = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} u(x') \, dx' \quad F : H \to \mathbb{R} \]

where \( \partial \Omega \) is bounding surface, \( |\partial \Omega| = \text{surface area} \) so that \( F \) is the average on \( \partial \Omega \).
EXAMPLE: Operators on $\ell^2 = H$

Let $x \in \ell^2$ be an (ordered) sequence $\{x_n\}$

$$x = \{x_1, x_2, x_3, \ldots \}$$

with standard $\ell^2$ inner product.

Define the shift operator

$$Lx = \{0, x_1, x_2, \ldots \}$$

Clearly $L : H \to H$. In particular $Lx \in \ell^2$ if $x \in \ell^2$.

$$\|Lx\|_2^2 = \sum_{i=1}^{\infty} x_i^2 = \|x\|_2^2$$

In fact $\|L^n x\|_2^2 = \|x\|_2^2$ by induction.

so

$$Lx \neq 0$$

as one might expect, noting

$$L^k x = \{0, 0, 0, \underbrace{0, \ldots 0}_{k}, x_1, x_2, \ldots \}$$
**EXAMPLE**  **Differential Operators**

Let \( H = L^2[a, b] \)

\[ D(L) = \{ u \in H : u \in C^2[a, b], u(a) = u(b) = 0 \} \]

is a linear subspace of \( H \). Then

\[ Lu = u'' + a_1(x)u' + a_0(x)u \]

where \( a_k \) are cont. on \([a, b]\).

The statement

\[ Lu = f \quad u \in D(L) \]

is a compact way of writing the

**Boundary Value Problem**

\[ (1) \quad u'' + a_1(x)u' + a_0(x)u = f(x) \]

\[ (2) \quad u(a) = 0, \quad u(b) = 0 \]

As we shall see, such operators can have nontrivial nullspaces. For instance

\[ Lu = u'' + n^2u \quad u(0) = u(\pi) = 0 \]

Then

\[ N(L) = \text{span} \{ \sin(nx) \} \]

since \( u \in N(L) \Rightarrow u'' + n^2u = 0 \).
**Example** Momentum/Energy operators

In quantum mechanics the "state" of a system is represented by an element of a Hilbert space $H$. One axiom is that every measurable quantity (or observable) has an operator $L$ associated with it.

Let $x \in \mathbb{R}^3$ be cartesian coordinates of a particle of mass $m$

$$\hat{p} = -i\hbar \nabla$$

is the momentum operator on $H = L^2(\mathbb{R}^3)$. Elements of $H$ are "wave functions" and

$$\int \frac{\psi(x)dx}{\hbar} = \text{prob particle in } \Omega$$

In (1), $\hbar$ is Planck's constant, $\nabla = \text{del operator}$, $i^2 = -1$.

In "classical" mechanics the energy of the particle under influence of a force of potential energy $V(x)$ is

$$E = \frac{1}{2m} \dot{x}^2 + V(x)$$

In quantum mechanics this corresponds to the "Hamiltonian" (operator on $H$)

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + V(x) = i \frac{\hbar^2}{2m} \nabla^2 + V(x)$$

Spectra of $\hat{H}$ are measurable energies:

$$\frac{\hbar^2}{2m} \nabla^2 \psi + V(x) \psi = E \psi$$
EXAMPLE  linear reaction systems (PDE)

Let \( H_1 \equiv L^2(\Omega) \) where \( \Omega \subset \mathbb{R}^3 \) compact.

\[
L \equiv H_1 \otimes H_1
\]

Then for \( u = (u_1, u_2) \in H \) we may define

\[
Lu = \begin{pmatrix}
\nabla^2 u_1 + a_{11} u_1 + a_{12} u_2 \\
\n\nabla^2 u_2 + a_{21} u_1 + a_{22} u_2
\end{pmatrix}
\]

sometimes written

\[
Lu = \nabla^2 u + Au
\]

for an appropriate matrix \( A \).

A common domain for such an operator is

\[
D(L) = \{ u \in H : \frac{\partial u}{\partial n} = 0, x \in \partial \Omega \}
\]

where \( \partial \Omega \) is the boundary of \( \Omega \) and \( \frac{\partial u}{\partial n} \) is the normal derivative.

The eigenvalue problem

\[
Lu = \lambda u \quad u \in D(L)
\]

comes up in the linear analysis of nonlinear "reaction diffusion" systems.
Nullspace of $L : H \rightarrow H$

$N(L) = \{ u \in D(L) : Lu = 0 \}$

**Example** Let $\Omega = [0, \pi]^2$ and $H = L^2(\Omega)$

$D(L) = \{ u \in H : u = 0, x \in \partial \Omega \}$

$Lu = \nabla^2 u + 25u$

Not hard to see

$u_{mn} = \sin(mx) \sin(ny) \in D(L)$

One can show $N(L)$ is dense in $H$

but there are elements of $H$ not pointwise equal.

Also not hard to show the following elements are in $N(L)$

$u_{34} = \sin(3x) \sin(4y)$

$u_{43} = \sin(4x) \sin(3y)$

Here

$N(L) = \text{span} \{ u_{34}, u_{43} \}$

$\dim N(L) = 2$.