Hilbert Spaces

Defn : A Hilbert space H is an inner product which is complete w.r.t. $\|x\| \equiv \langle x, x \rangle^{1/2}$

Below we give several examples of Hilbert Spaces

EXAMPLE

$$H = L^2(\Omega) \qquad \langle f, g \rangle \equiv \int_\Omega f \bar{g} \, dx$$

with the understanding $f = g$ is in the a.e. Lebesgue sense

EXAMPLE

$$H = L^2(\Omega) \qquad \langle f, g \rangle \equiv \int_\Omega r f \bar{g} \, dx$$

Providing r is integrable and

$$0 < r(x) \leq M \qquad \forall x \in \Omega$$

EXAMPLE

$$H = \ell^2 = \left\{ \{x_n\} : x_k \in \mathbb{C}, \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$$

Here the inner product is

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n$$

where x, y are sequences.

All above stated without proof of completeness.

EXAMPLE (Sobolev Spaces)

$$\langle f, g \rangle \equiv \int_a^b \sum_{j=0}^{n} f^{(j)}(x)\, \overline{g^{(j)}(x)}\, dx$$

is an inner product subspace of $L^2[a,b]$. Following is a Hilbert space

$$H^n[a,b] \equiv \{ f \in L^2[a,b] : \| f \| < \infty \}$$

Note that $H^n[a,b]$ is a proper subset of $L^2[a,b]$ for $n \geq 1$.

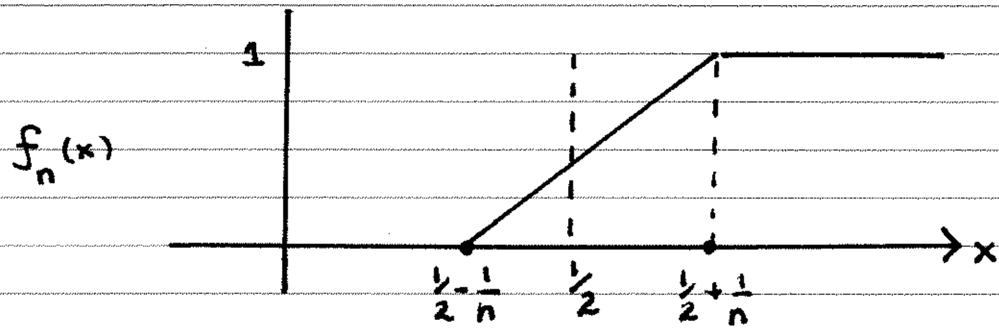
EXAMPLE Let $C_0^\infty(\Omega)$ be set of all C^∞ functions of compact support.

The completion of $C_0^\infty(\Omega)$ in $H^n[a,b]$ is defined as $\overset{\circ}{H}{}^n(\Omega)$ and is a Hilbert space.

EXAMPLE <u>Non complete inner product space</u>

$$\mathbb{X} \equiv \{ f \in C[0,1] : \| f \| < \infty \}$$

using standard L^2 inner product



Explicit formula for $\{f_n\}$ sequence

$$f_n(x) = \begin{cases} 0 & x \in (0, \frac{1}{2}-\frac{1}{n}) \\ \frac{1}{2}+\frac{n}{2}(x-\frac{1}{2}) & x \in (\frac{1}{2}-\frac{1}{n}, \frac{1}{2}+\frac{1}{n}) \\ 1 & x \in (\frac{1}{2}+\frac{1}{n}, 1) \end{cases}$$

Very clear that

$$f_n(x) \longrightarrow H(x-\tfrac{1}{2}) \quad \notin \mathbb{X}$$

i.e, a Cauchy sequence that does not converge to an element of $\mathbb{X}$

<u>Remark</u> Recall convergence implies Cauchy. $\exists N$ s.t.

$$\| f_n - f_m \| \leq \| f_n - f \| + \| f_m - f \| \leq 2\varepsilon$$

$\forall n, m \geq N$ since

$$\| f_n - f \| \leq \varepsilon \qquad \forall n \geq N$$

if $f_n \to f$.

Banach versus Hilbert spaces

A Banach space is a complete normed space.

$$\mathbb{X} \equiv \left\{ f \in C[a,b] : \|f\| < \infty, \ \|f\| \equiv \max_{[a,b]} |f(x)| \right\}$$

is a linear space with a norm not induced by a metric. It is also complete (below) hence is a Banach space.

However, $\mathbb{X}' \equiv C[a,b]$ with the L^2 inner product is not a Hilbert space. There are many continuous $\{f_n\}$ where $f_n \to f \notin \mathbb{X}'$. Hence, the unidirectional implication

$$\text{Hilbert} \Rightarrow \text{Banach}$$

Completeness proof

Let $\{f_n\} \subset \mathbb{X}$ be Cauchy in $\mathbb{X}$. Then for each $x \in [a,b]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ complete, $\exists f(x)$ s.t.

$$f_n(x) \to f(x) \qquad \text{pointwise}$$

To show $f(x) \in \mathbb{X}$, i.e. continuous:

$$|f(t) - f(t+\delta)| \le \underset{\downarrow \atop 0}{|f(t) - f_K(t)|} + \underbrace{|f_K(t) - f_K(t+\delta)|}_{\text{cont}^y \text{ of } f_K(t)} + \underset{\downarrow \atop 0}{|f_K(t+\delta) - f(t+\delta)|}$$

Approximation by finite sets (same as before)

Let H be a real Hilbert space with $\langle x,y\rangle = \langle y,x\rangle$

Let $T = \{\phi_k\}_{k=1}^{n} \subset H$ and $f \in H$. Here the functions ϕ_k need not be orthogonal

$$(1) \qquad F(\alpha) \equiv \| f - \sum_{k=1}^{n} \alpha_k \phi_k \|^2$$

As before $\frac{\partial F}{\partial \alpha_k} = 0$ yields a system for α_k:

$$(2) \qquad \Phi \alpha = \beta$$

where $\alpha = (\alpha_1 \ldots \alpha_n)^T$ and

$$[\Phi]_{ij} = \langle \phi_i, \phi_j \rangle \qquad \beta_j = \langle f, \phi_j \rangle$$

If ϕ_k are independent, eqn (2) has a unique soln. Defn of ind.

$$\sum_{i=1}^{n} \alpha_i \phi_i = 0 \qquad \Leftrightarrow \alpha_i = 0 \ \forall i$$

$$\sum_{i=1}^{n} \alpha_i \langle \phi_i, \phi_j \rangle = 0 \qquad \Leftrightarrow \alpha_i = 0 \ \forall i$$

$$\Phi \alpha = 0 \qquad \Leftrightarrow \alpha = 0$$

The solution of (2) yields the least squares approximation relative to the independent set T. The approximator is in span(T).

EXAMPLE $H = L^2(-1,1)$ and $f(x) \equiv 2H(x) - 1$ where $H(x)$ is the Heaviside fn.

$$\phi_i(x) \equiv x^{2i-1} \qquad i = 1, 2, 3$$

are odd monomials. Least squares approximation

$$\bar{f}(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x)$$

where α a solution of $\Phi \alpha = \beta$

(self adjoint) $\Phi_{ij} = \langle \phi_i, \phi_j \rangle \qquad \beta_i = \langle \phi_i, f \rangle$

For $n = 1, 2, 3$ can compute

$n=1 \qquad \Phi = [2/3] \qquad \beta = [1]$

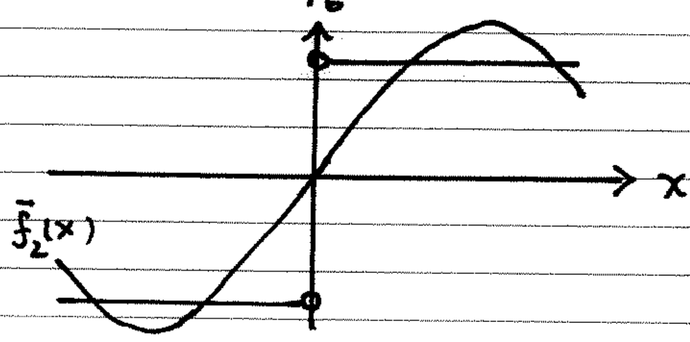
$n=2 \qquad \Phi = \begin{bmatrix} 2/3 & 2/5 \\ 2/5 & 2/7 \end{bmatrix} \qquad \beta = [1, 1/2]^T$

$n=3 \qquad \Phi = \begin{bmatrix} 2/3 & 2/5 & 2/7 \\ 2/5 & 2/7 & 2/9 \\ 2/7 & 2/9 & \frac{2}{11} \end{bmatrix} \qquad \beta = [1, 1/2, 1/3]^T$

For example solving $\Phi \alpha = \beta$ for $n = 1, 2$ we find α:

$n=1 \qquad \bar{f}_1(x) = \frac{3}{2} x$

$n=2 \qquad \bar{f}_2(x) = \frac{5}{16}(9x - 7x^3)$



General notes

$$\langle \phi_i, \phi_j \rangle = \frac{2}{2(i+j)-1}$$

Approximation by finite orthonormal sets

As before

$$F(\alpha) \equiv \| f - \sum_{k=1}^{n} \alpha_k \phi_k \|^2$$

is minimized by any α that satisfies

(1) $$\Phi \alpha = \beta \qquad \Phi_{ij} = \langle \phi_i, \phi_j \rangle \quad \beta_i = \langle \phi_i, f \rangle$$

If we now assume $\{\phi_i\}$ is an orthonormal set then

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} \qquad \text{(Kronecker delta)}$$

in which case $\Phi = I$, the identity matrix.

$$\boxed{f(x) \approx \bar{f}_n(x) \equiv \sum_{i=1}^{n} \langle \phi_i, f \rangle \phi_i}$$

In this context

$\langle \phi_i, f \rangle$ = Fourier coefficients

$\bar{f}_n(x)$ = Fourier series (generalized)

<u>Key issue</u>: does the series converge to $f(x)$?

Approximation Error, and Bessel's Inequality

Let $\{\phi_i\}$ be an orthonormal set on some Hibert space H.

$$E(\alpha) \equiv \| f - \sum_{i=1}^{n} \alpha_i \phi_i \|^2 \qquad \alpha_i \equiv \langle \phi_i, f \rangle$$

$$E(\alpha) = \langle f - \sum \alpha_i \phi_i , f - \sum \alpha_j \phi_j \rangle$$

$$E(\alpha) = \|f\|^2 - 2 \sum \alpha_i \langle \phi_i, f \rangle + \sum_i \sum_j \alpha_i \alpha_j \underbrace{\langle \phi_i, \phi_j \rangle}_{\delta_{ij}}$$

$$E(\alpha) = \|f\|^2 - 2 \sum \alpha_i^2 + \sum \alpha_i^2$$

$$E(\alpha) = \|f\|^2 - \sum_{i=1}^{n} \langle f, \phi_i \rangle^2$$

Summarize

$$\boxed{E(\alpha) = \|f\|^2 - \sum_{i=1}^{n} \langle f, \phi_i \rangle^2 \geq 0}$$

Given $E(\alpha) \geq 0$ we then have Bessel's Inequality

$$\boxed{\sum_{i=1}^{n} \langle f, \phi_i \rangle^2 \leq \|f\|^2 < \infty, \quad \forall n}$$

this series converges as $n \to \infty$

Fourier Series Convergence

Let $\{\phi_i\}_{i=1}^{\infty}$ be orthonormal set in H

Fourier series

$$\bar{f}_n \equiv \sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i \qquad \alpha_i = \langle f, \phi_i \rangle$$

Claim $\{\bar{f}_n\}$ Cauchy in H. wlog $n > m$

$$\| \bar{f}_n - \bar{f}_m \|^2 = \left\langle \sum_{i=1}^{n} \alpha_i \phi_i , \sum_{j=1}^{m} \alpha_j \phi_j \right\rangle , \quad \langle \phi_i, \phi_j \rangle = \delta_{ij}$$

$$= \left\| \sum_{m+1}^{n} \langle f, \phi_i \rangle^2 \phi_i \right\|^2 \qquad \text{cross terms out}$$

$$= \sum_{m+1}^{n} \langle f, \phi_i \rangle^2 \qquad \|\phi_i\| = 1$$

$$\leq \sum_{m+1}^{\infty} \langle f, \phi_i \rangle^2$$

The infinite sum exists on account of Bessel's inequality. Moreover, it can be made small, i.e.

$$\| \bar{f}_n - \bar{f}_m \|^2 \leq \sum_{m+1}^{\infty} \langle f, \phi_i \rangle^2 \leq \varepsilon , \quad \forall m,n \geq N$$

Since $\{\bar{f}_n\}$ is Cauchy in a complete space H there exists an $\bar{f}$ such that $\bar{f}_n \to \bar{f}$.

Incomplete orthonormal sets.

Not all countable orthonormal sets $\{\phi_n\}$ form a basis for H.

$$\text{span}\{\phi_n\} \neq H$$

This is best seen by way of examples

EX $H \equiv L^2[0, 2\pi]$ and $\{\phi_n(x)\} = \{\frac{1}{\sqrt{\pi}} \sin(nx)\}$

One can verify

$$\langle \phi_n, \phi_m \rangle = \int_0^{2\pi} \sin(mx)\sin(nx)\,dx = 0, \quad m \neq n$$

$$\langle \phi_n, \phi_n \rangle = \int_0^{2\pi} \frac{1}{\pi}\sin^2(nx)\,dx = 1$$

But for $f(x) = 1$ we have

$$\langle \phi_n, f \rangle = \int_0^{2\pi} 1 \cdot \sin(nx)\,dx = 0$$

so that the Fourier series

$$f_n(x) = \sum_{k=1}^{n} \langle \phi_k, f \rangle \phi_k = 0 \qquad \forall n$$

Here $f_n(x) \to 0 \neq f(x)$.

EXAMPLE Legendre polynomials $\{P_n(x)\}$

Can be found by using Gram Schmidt of monomials $\{x^n\}$ on $L^2[-1,1]$.

Can be shown (not here) that $\{P_n(x)\}$ is an orthogonal (not normalized) spanning set for $L^2[-1,1]$. They are solutions to Legendre's differential equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + n(n+1)u = 0$$

Such ODE's arise in physical problems involving spheres.

Given $T = \{P_n(x)\}_{n\geq 0}$ we define

$$T' = \{P_n(x)\}_{n=2m+1}$$

odd power polynomials in T.

If we attempted a Fourier series of $f(x)=1$ using $T' \subset T$ we'd have

$$\langle 1, P_n(x)\rangle = \int_{-1}^{1} P_n(x)\,dx = 0$$

$f(x)$ even
$P_n(x)$ odd

So that

$$f_n(x) = \sum_{m=0} \langle P_{2m+1}, f\rangle P_{2m+1} \equiv 0 \qquad \forall n$$

Again $f_n(x) \to 0 \neq f(x) \equiv 1$.

<u>Defn</u> An orthonormal set $\{\phi_n\}_{n=1}^{\infty}$ is <u>complete</u> if

$$f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \qquad \forall f \in H$$

<u>Theorem</u> The following are equivalent for orthnorm set $\{\phi_n\}$:

(1) $\{\phi_n\}_{n=1}^{\infty}$ is a complete orthonormal set for H

(2) $f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \qquad \forall f \in H$

(3) $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \qquad \forall f \in H$ (Parseval's equality)

(4) $\langle f, \phi_n \rangle = 0 \;\; \forall n \quad \Rightarrow \quad f = 0$

<u>Proof (2) $\Rightarrow$ (3)</u> Due to Bessels inequality $\sum_{n=1} \langle f, \phi_n \rangle^2 < \infty$.

$$\begin{aligned} \|f\|^2 &= \left\langle \sum_n \langle f, \phi_n \rangle \phi_n , \sum_m \langle f, \phi_m \rangle \phi_m \right\rangle \\ &= \sum_n \sum_m \langle f, \phi_n \rangle \overline{\langle f, \phi_m \rangle} \langle \phi_n, \phi_m \rangle \\ &= \sum_n |\langle f, \phi_n \rangle|^2 \end{aligned}$$

<u>Proof (3) $\Rightarrow$ (4)</u> If $\langle f, \phi_n \rangle = 0 \;\; \forall n$, Parseval's equality immediately gives

$$\|f\|^2 = 0 \qquad \Rightarrow \qquad f = 0.$$

See Naylor/Sell pg 307 (Linear Operator Theory in Engineering and Science) for complete proofs.

Operators on Hilbert Spaces

Given some Hilbert space any mapping

$$L : H \to H$$

is an operator. In general the domain of L need not be equal to H

$$D(L) \subset H \qquad \text{domain}$$

$$R(L) \subset H \qquad \text{range}$$

We will also restrict our attention to linear operators

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 L(u_1) + \alpha_2 L(u_2)$$

for all $\alpha_k \in \mathbb{C}$ and $u_k \in D(L)$

EXAMPLE $H = \mathbb{R}^n$ (matrix multiply)

$$Lu = Au \qquad A \in \mathbb{R}^{n \times n}$$

here $D(L) = \mathbb{R}^n$.

EXAMPLE Integral operators

Let $\Omega = [a,b]$ and define

$$Ku = \int_a^b k(x,y)\,u(y)\,dy$$

for some continuous $k(x,y)$ "kernel"

For example

$$Ku \equiv \int_0^1 \log(x+y)\, u(y)\, dy$$

For $u(y) \equiv 1$

$$K(u) = \int_0^1 \log(x+y)\, dy$$

can be computed using IBP

$$K(u) = (x+1)\ln(x+1) - (x+1)$$

For $x \in [0,1]$ this is well defined and in $H = L^2[0,1]$.

<u>EXAMPLE</u> More general integral operators

$$\Omega \subset \mathbb{R}^n \qquad \text{compact}$$

$$K(u) = \int_\Omega k(x,x')\, u(x')\, dx'$$

For appropriate k, $K : L^2(\Omega) \to L^2(\Omega)$

Should be contrasted with a <u>functional</u> on a Hilbert Space such as

$$F(u) = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u(x')\, dx' \qquad F : H \to \mathbb{R}$$

where $\partial\Omega$ is bounding surface, $|\partial\Omega|$ = surface area so that F is the average on $\partial\Omega$

EXAMPLE Operators on $\ell^2 = H$

Let $x \in \ell^2$ be an (ordered) sequence $\{x_n\}$

$$x = \{x_1, x_2, x_3, \ldots\}$$

with standard ℓ^2 inner product.

Define the shift operator

$$Lx \equiv \{0, x_1, x_2, \ldots\}$$

Clearly $L: H \to H$. In particular $Lx \in \ell^2$ if $x \in \ell^2$.

$$\|Lx\|^2 = \sum_{i=1}^{\infty} x_i^2 = \|x\|^2$$

In fact $\|L^n x\|^2 = \|x\|^2$ by induction so

$$Lx \not\to 0$$

as one might expect, noting

$$L^k x = \{\underbrace{0, 0, 0, 0, \ldots 0}_{k}, x_1, x_2, \ldots\}$$

EXAMPLE Differential Operators

Let $H = L^2[a,b]$

$$D(L) \equiv \{ u \in H \ : u \in C^2[a,b],\ u(a) = u(b) = 0 \}$$

is a linear subspace of H. Then

$$Lu \equiv u'' + a_1(x)u' + a_0(x)u$$

where a_K are cont. on $[a,b]$.

The statement

$$Lu = f \qquad u \in D(L)$$

is a compact way of writing the Boundary Value Problem

$$(1) \qquad u'' + a_1(x)u' + a_0(x)u = f(x)$$

$$(2) \qquad u(a) = 0 \ , \quad u(b) = 0$$

As we shall see, such operators can have nontrivial nullspaces. For instance

$$Lu \equiv u'' + n^2 u \qquad u(0) = u(\pi) = 0$$

Then

$$N(L) = \text{span}\{ \sin(nx) \}$$

since $v \in N(L) \Rightarrow v'' + n^2 v = 0$.

EXAMPLE Momentum / Energy operators

In quantum mechanics the "state" of a system is represented by an element of a Hilbert space H. One axiom is that every measurable quantity (or observable) has an operator L associated with it.

Let $x \in \mathbb{R}^3$ be cartesian coordinates of a particle of mass m

$$(1) \qquad \hat{p} \equiv -i\hbar \nabla$$

is the momentum operator on $H = L^2(\mathbb{R}^3)$
Elements of H are "wave functions" and

$$\int_\Omega \psi(x)\,dx = \text{prob particle in } \Omega$$

In (1), $\hbar$ is Planck's const, ∇ = del operator, $i^2 = -1$.

In "Classical" mechanics the energy of the particle under influence of a force of potential energy $V(x)$ is

$$(2) \qquad E = \frac{1}{2m} p^2 + V(x) \qquad \text{total energy}$$

In quantum mechanics this corresponds to the "Hamiltonian" (operator on H)

$$(3) \qquad \hat{H} \equiv \frac{1}{2m}\hat{p}^2 + V(x) = i\frac{\hbar^2}{2m}\nabla^2 + V(x)$$

Spectra of $\hat{H}$ are measurable energis:

$$(4) \qquad i\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = E\psi$$

EXAMPLE Linear reaction systems (PDE)

Let $H_1 \equiv L^2(\Omega)$ where $\Omega \subset \mathbb{R}^3$ compact.

$$H \equiv H_1 \otimes H_1$$

Then for $u = (u_1, u_2) \in H$ we may define

$$Lu = \begin{pmatrix} \nabla^2 u_1 + a_{11} u_1 + a_{12} u_2 \\ \nabla^2 u_2 + a_{21} u_1 + a_{22} u_2 \end{pmatrix}$$

sometimes writter

$$Lu = \nabla^2 u + Au$$

for an appropriate matrix A.

A common domain for such an operator is

$$D(L) = \left\{ u \in H : \frac{\partial u_k}{\partial n} = 0 \, , \, x \in \partial\Omega \right\}$$

where $\partial\Omega$ is the boundary of Ω and $\frac{\partial u_k}{\partial n}$ is the normal derivative.

The eigenvalue problem

$$Lu = \lambda u \qquad u \in D(L)$$

comes up in the linear analysis of nonlinear "reaction diffusion" systems.

Nullspace of $L : H \to H$

$$N(L) \equiv \{ u \in D(L) : Lu = 0 \}$$

EXAMPLE Let $\Omega = [0, \pi]^2$ and $H = L^2(\Omega)$

$$D(L) = \{ u \in H : u = 0 , x \in \partial\Omega \}$$

$$Lu \equiv \nabla^2 u + 25 u$$

Not hard to see

$$u_{mn} \equiv \sin(mx)\sin(ny) \quad \in D(L)$$

One can show N(L) is dense in H
but there are elements of H not
pointwise equal.

Also not hard to show the following
elements are in N(L)

$$u_{34} = \sin(3x)\sin(4y)$$

$$u_{43} = \sin(4x)\sin(3y)$$

Here

$$N(L) = \text{span}\{ u_{34}, u_{43} \}$$

$$\dim N(L) = 2.$$