# Differential Operator inverse

Many such problems can be posed

(1) $\qquad Lu = f \qquad\qquad u \in D(L)$

Providing $N(L) = 0$ we may seek an inverse (or solution) for the problem.

(2) $\qquad u = L^{-1} f \qquad\qquad u \in D(L)$

A concrete example is

(3) $\qquad Lu = u''$

(4) $\qquad D(L) = \{ u \in C^2[0,1] : u(0) = u(1) = 0 \}$

one may show (later) that for this $L$ and boundary conditions

$$u'' = f \qquad\qquad u \in D(L)$$

has the solution

(5) $$u(x) = \int_0^1 g(x,y) f(y)\, dy \equiv \mathbb{K} f$$

where

$$g(x,y) = \begin{cases} y(x-1) & 0 \le y < x \le 1 \\ x(y-1) & 0 \le x < y \le 1 \end{cases}$$

The kernel is a Green's function.

To reiterate the solution of

$$Lu \equiv u'' = f \qquad u \in D(L)$$

is the integral operator

$$u = L^{-1} f = \underline{K} f \equiv \int_0^1 g(x,y) f(y)\, dy$$

$\underline{K}$ being compact / self adjoint has a complete set of orthonormal eigenfunctions

$$\underline{K} \phi_n = \mu_n \phi_n \qquad \phi_n \in D(L)$$

$$L^{-1} \phi_n = \mu_n \phi_n$$

$$L L^{-1} \phi_n = \mu_n L \phi_n$$

$$\phi_n = \mu_n L \phi_n$$

or,

$$\boxed{L \phi_n = \frac{1}{\mu_n} \phi_n}$$

Alternately $L$ has the same set of e-fns but with eigenvalues

$$\lambda_n = \frac{1}{\mu_n}$$

and $|\lambda_n| \to \infty$ since $|\mu_n| \to 0$.

Appendix : Want to show that if

$$u(x) = \int_0^1 g(x,y)\, f(y)\, dy$$

then

$$u''(x) = f(x)$$

for the stated Green's function

$$u(x) = \int_0^1 g(x,y)\, f(y)\, dy$$

$$u(x) = \int_0^x y(x-1) f(y)\, dy + \int_x^1 x(y-1) f(y)\, dy$$

Differentiate in x using Liebniz's rule.

$$u'(x) = \cancel{x(x-1)f(x)} + \int_0^x y\, f(y)\, dy$$

$$\qquad - \cancel{x(x-1)f(x)} + \int_x^1 (y-1) f(y)\, dy$$

$$u'(x) = \int_0^x y\, f(y)\, dy + \int_x^1 (y-1) f(y)\, dy$$

$$u''(x) = x f(x) - (x-1) f(x)$$

$$u''(x) = f(x)$$

## Inverse Operators are linear

Let $L : H \to H$ be a linear operator and

(1) $$L u_k = f_k \qquad k = 1, 2$$

hence

(2) $$u_k = L^{-1} f_k$$

Summing over $k$ in (2)

(3) $$u_1 + u_2 = L^{-1} f_1 + L^{-1} f_2$$

But linearity of $L$ in (1) implies

(4) $$L(u_1 + u_2) = f_1 + f_2$$

(5) $$u_1 + u_2 = L^{-1}(f_1 + f_2)$$

Comparing to (3) we have

$$L^{-1}(f_1 + f_2) = L^{-1} f_1 + L^{-1} f_2$$

Minor modifications yield

$$L^{-1}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 L^{-1} f_1 + \alpha_2 L^{-1} f_2$$

$\forall \alpha_k \in \mathbb{C}$.

## Connection to resolvents of compact operators

Let $L^{-1}$ be well defined inverse of

$$(1) \qquad Lu = f \qquad u \in D(L)$$

An associated problem

$$(2) \qquad (L - \lambda I)u = f \qquad u \in D(L)$$

Note that if $\lambda$ is an eigenvalue of $L$ then $N(L-\lambda I) \neq 0$ and (2) can't have a unique inverse, i.e. $(L-\lambda I)^{-1}$ D.N.E.

Assuming $\lambda$ not an eigenvalue:

$$L^{-1}(L - \lambda I)u = L^{-1}f$$

$$L^{-1}Lu - \lambda L^{-1}u = L^{-1}f \qquad g \equiv L^{-1}f$$

$$Iu - \lambda L^{-1}u = g$$

$$\boxed{(I - \lambda \underline{K})u = g}$$

where $\underline{K}$ is the integral operator inverse of the differential operator $L$, say.

## Greens functions, inverses and the $\delta$-function

The solution of

(1) $$u'' = f(x) \qquad u(0) = u(1) = 0$$

is given by

$$u(x) = L^{-1} f = \int_0^1 g(x,y)\, f(y)\, dy$$

where

$$g(x,y) = \begin{cases} y(x-1) & 0 \le y < x \le 1 \\ x(y-1) & 0 \le x < y \le 1 \end{cases}$$

For all $u \in D(L)$ one would expect $u = L(L^{-1}u)$

(2) $$u(x) = L\left( \int_0^1 g(x,y)\, u(y)\, dy \right)$$

$$u(x) = \int_0^1 Lg \cdot u\, dy$$

(3) $$u(x) = \int_0^1 \delta(x,y)\, u(y)\, dy \qquad \forall u \in D(L)$$

There is no function $\delta(x,y) \in C[0,1]^2$ such that (3) is true for all $u \in D(L)$.

The mathematical mistake occurs at the indicated (arrowed) step. Eqn (2) <u>is</u> well defined though.

The theory of distributions tries to make sense of (3) use a Dirac delta function $\delta(y-x)$ in the sense that $g$ is a solution of the distributional equation

$$(4) \qquad Lg = \delta(y-x) \qquad g \in D(L)$$

so that

$$(5) \qquad u(x) = \int_0^1 \delta(y-x)\, u(y)\, dy \qquad \forall u \in D(L)$$

Again this doesn't make sense as a "function".

The precise mean of the RHS of (5) is that $\delta$ is a "distribution":

$$\delta : H \longrightarrow \mathbb{R}$$

where $\delta$ is defined by

$$\langle \delta, u \rangle = u(x)$$

that is, $\delta$ evaluates $u$ at $x$. $u(x) \in \mathbb{R}$.

## Broad Overview

We wish to solve the problem

$$(2) \qquad Lu = f \qquad\qquad u \in D(L)$$

We assume $L$ has an adjoint

$$(3) \qquad \langle Lu, v\rangle = \langle u, L^* v\rangle \qquad\qquad \forall v \in D(L^*)$$

To solve (2) we seek a Green's function s.t.

$$(4) \qquad L^* g = \delta(y-x) \qquad\qquad g \in D(L^*)$$

and $\delta$ is the delta "dist$^n$".

The way we make sense of (4) is to define distn's on test function spaces $D$. Typically $D = C^\infty(\Omega)$ of compact support. Then

$$(5a) \qquad \langle L^* g, \phi\rangle = \phi(x) \qquad\qquad \text{Dist}^n$$

$$(5b) \qquad \langle g, L\phi\rangle = \phi(x) \qquad\qquad L^2 \text{ sense}$$

If there is a $g(x,y)$ function satisfying (5b):

$$\langle Lu, g\rangle = \langle u, L^* g\rangle$$

$$\langle f, g\rangle = u(x)$$

or

$$u(x) = \int_\Omega g(x,y) f(y)\, dy$$