Differential Operator inverse

Many such problems can be posed

\[(1) \quad Lu = f \quad u \in D(L)\]

Providing \(N(L) = 0\) we may seek an inverse \(L^{-1}\) (or solution) for the problem.

\[(2) \quad u = L^{-1}f \quad u \in D(L)\]

A concrete example is

\[(3) \quad Lu = u''\]

\[(4) \quad D(L) = \{ u \in C^2[0,1] : u(0) = u(1) = 0 \}\]

one may show (later) that for this \(L\) and boundary conditions

\[u'' = f \quad u \in D(L)\]

has the solution

\[u(x) = \int_0^1 g(x,y)f(y)\,dy \equiv Kf\]

where

\[g(x,y) = \begin{cases} y(x-1) & 0 \leq y < x < 1 \\ x(y-1) & 0 \leq x < y < 1 \end{cases}\]

The kernel is a Green's function.
To reiterate the solution of
\[ Lu = u'' = f \quad \text{u} \in \text{D}(L) \]
is the integral operator
\[ u = L^{-1}f = Kf = \int_0^1 g(x,y)f(y)dy \]
\( K \) being compact/self-adjoint has a complete set of orthonormal eigenfunctions
\[ K \phi_n = \mu_n \phi_n \quad \phi_n \in \text{D}(L) \]
\[ L^{-1} \phi_n = \mu_n \phi_n \]
\[ L L^{-1} \phi_n = \mu_n L \phi_n \]
\[ \phi_n = \mu_n L \phi_n \]
or,
\[ L \phi_n = \frac{1}{\mu_n} \phi_n \]
Alternatively \( L \) has the same set of \( e \)-fns but with/eigenvalues
\[ \lambda_n = \frac{1}{\mu_n} \]
and \( |\lambda_n| \to \infty \) since \( |\mu_n| \to 0 \).
\textbf{Appendix} : Want to show that if
\[ u(x) = \int g(x,y) f(y) \, dy \]
then
\[ u''(x) = f(x) \]
for the stated Green's function
\[ u(x) = \int g(x,y) f(y) \, dy \]
\[ u(x) = \int_0^x g(x,y) f(y) \, dy + \int_x^1 g(x,y-1) f(y) \, dy \]

Differentiate in \( x \) using \textit{Liebnitz's rule}.
\[ u'(x) = \int_0^x (x-y)f(y) \, dy + \int_x^1 yf(y) \, dy \]
\[ -x(x-1)f(x) + \int_0^1 (y-1)f(y) \, dy \]
\[ u'(x) = \int_0^x yf(y) \, dy + \int_x^1 (y-1)f(y) \, dy \]
\[ u''(x) = x f(x) - (x-1) f(x) \]
\[ u''(x) = f(x) \]
Inverse Operators are linear

Let $L : H \to H$ be a linear operator and

\[ L u_k = f_k, \quad k = 1, 2 \]

hence

\[ u_k = L^{-1} f_k \]

Summing over $k$ in (2)

\[ u_1 + u_2 = L^{-1} f_1 + L^{-1} f_2 \]

But linearity of $L$ in (1) implies

\[ L(u_1 + u_2) = f_1 + f_2 \]

Comparing to (3) we have

\[ L^{-1} (f_1 + f_2) = L^{-1} f_1 + L^{-1} f_2 \]

Minor modifications yield

\[ L^{-1} (\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 L^{-1} f_1 + \alpha_2 L^{-1} f_2 \]

∀\( u_k \in C \).
Connection to resolvents of compact operators

Let $L^{-1}$ be well defined inverse of

1) $Lu = f \quad u \in D(L)$

An associated problem

2) $(L - \lambda I)u = f \quad u \in D(L)$

Note that if $\lambda$ is an eigenvalue of $L$ then $N(L - \lambda I) \neq 0$ and (2) can't have a unique inverse, i.e., $(L - \lambda I)^{-1}$ D.N.E.

Assuming $\lambda$ not an eigenvalue:

$L^{-1}(L - \lambda I)u = L^{-1}f$  
$L^{-1}Lu - L^{-1}\lambda u = L^{-1}f$  
$Iu - \lambda L^{-1}u = g$  
$(I - \lambda K)u = g$

where $K$ is the integral operator inverse of the differential operator $L$, say.
Greens functions, inverses and the $\delta$-function

The solution of

\begin{equation}
u'' = f(x) \quad u(0) = u(1) = 0
\end{equation}

is given by

\[ u(x) = L^{-1}f = \int_0^1 g(x,y)f(y)\,dy \]

where

\[ g(x,y) = \begin{cases} 
  \frac{1}{x-1} & 0 \leq y < x \leq 1 \\
  \frac{1}{y-1} & 0 \leq x < y \leq 1
\end{cases} \]

For all $u \in D(L)$ one would expect $u = L(L^{-1}u)$

\begin{equation}
(2) \quad u(x) = L\left(\int_0^1 g(x,y)u(y)\,dy\right)
\end{equation}

\[ u(x) = \int_0^1 g \cdot u \,dy \]

\begin{equation}
(3) \quad u(x) = \int_0^1 \delta(x,y)u(y)\,dy \quad \forall u \in D(L)
\end{equation}

There is no function $\delta(x,y) \in C[0,1]^2$ such that (3) is true for all $u \in D(L)$.

The mathematical mistake occurs at the indicated (arrowed) step. Eqn (2) is well defined though.
The theory of distributions tries to make sense of (3) use a Dirac delta function $\delta(y-x)$ in the sense that $q$ is a solution of the distributional equation

\[ L q = \delta(y-x) \quad q \in D(L) \]

so that

\[ u(x) = \int_0^1 \delta(y-x) u(y) \, dy \quad \forall u \in D(L) \]

Again this doesn't make sense as a "function". The precise mean of the RHS of (5) is that $\delta$ is a "distribution":

$\delta : H \rightarrow \mathbb{R}$

where $\delta$ is defined by

$\langle \delta, u \rangle = u(x)$

that is, $\delta$ evaluates $u$ at $x$. $u(x) \in \mathbb{R}$. 
Broad Overview

We wish to solve the problem

(2) \[ Lu = f \quad u \in D(L) \]

We assume \( L \) has an adjoint

(3) \[ \langle Lu, v \rangle = \langle u, L^*v \rangle \quad \forall u \in D(L^*) \]

To solve (2) we seek a Green's function s.t.

(4) \[ L^*g = \delta(y-x) \quad g \in D(L^*) \]

and \( \delta \) is the delta "distn".

The way we make sense of (4) is to define distns on test function spaces \( D \). Typically \( D = C^\infty_c(\mathbb{R}) \) of compact support.

Then

(5a) \[ \langle L^*g, \phi \rangle = \phi(x) \quad \text{Distr} \]

(5b) \[ \langle g, L\phi \rangle = \phi(x) \quad \text{\( L^2 \) sense} \]

If there is a \( g(x,y) \) function satisfying (5b):

\[ \langle Lu, g \rangle = \langle u, L^*g \rangle \]

\[ \langle f, g \rangle = u(x) \]

or

\[ u(x) = \int g(x,y) f(y) \, dy \]