Linear Algebra Review

Defn: A vector space S is any linear space closed under vector addition and scalar multiplication

Examples

1) $S = \{ \vec{x} \in \mathbb{R}^3 : \vec{x} = \alpha(1,3,2) \text{ for some } \alpha \in \mathbb{R} \}$

2) $S = \{ f(x) : f : \mathbb{R} \to \mathbb{R},\ f(x) = \sum_{i=0}^{N} \alpha_i x^i \} \equiv P_N$

3) $S = \{ f(x) : f \in P_N,\ f(0) = 0 \}$

4) $S = \{ u(x) : u'' + u = 0 \}$

a couple examples of spaces that are not vector spaces

5) $S = \{ f(x) : f \in P_N,\ f(2) = 7 \}$

6) $S = \{ u(x) : u' + u^2 = 0 \}$

Defn: A set $\{x_i\}_{i=1}^{n} \subset S$ (vector space) is independent iff

$$\sum_{k=1}^{n} \alpha_k x_k = 0 \iff \alpha_k = 0,\ \forall k$$

Defn: $T \subset S$ is a spanning set if every $x \in S$ can be written as a linear combination of elements of T.

Remark: In both these two defns one can have $n = \infty$ for the linear combinations

Defn: A <u>basis</u> is a linearly independent spanning set $\mathbb{T} \subset S$. For finite $\mathbb{T}$ we have

$$\dim(S) = \#\text{ elements in } \mathbb{T}$$

Defn: S is an <u>inner product space</u> if it is a vectorspace with a bilinear operator (inner product) $\langle \cdot , \cdot \rangle : S \times S \to \mathbb{C}$ such that

a) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

b) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \qquad \forall \alpha \in \mathbb{C}$

c) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(real) d) $\langle x, x \rangle \geq 0 \qquad \forall x \in S$

e) $\langle x, x \rangle = 0 \iff x = 0$

<u>EXAMPLES</u>

1) $S = \mathbb{R}^n \qquad \langle x, y \rangle \equiv \sum_{i=1}^{n} x_i \overline{y_i}$ Euclidean

2) $S = L^2[a,b] \qquad \langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\,dx$ L^2

and a Sobolev space $H^1[a,b]$

3) $S = H^1[a,b] \qquad \langle f, g \rangle = \int_a^b f(x)\overline{g(x)} + f'(x)\overline{g'(x)}\,dx$

Here $H^1[a,b]$ is the set of $f \in L^2[a,b]$ such that $\langle f, f \rangle$ is defined in 3).

Defn Every inner product space S has an <u>inner product induced norm</u>

$$\|x\| = \langle x, x\rangle^{1/2}$$

with the properties

a) $\|x\| \geq 0$

b) $\|x\| = 0 \iff x = 0$

c) $\|\alpha x\| = |\alpha| \, \|x\|$

d) $\|x+y\| \leq \|x\| + \|y\|$

Defn Let S be an inner product space. $x, y \in S$ <u>orthogonal</u> $\iff \langle x, y\rangle = 0$

<u>Examples</u>

1) $x = (1,2,1)$, $y = (1,-1,1)$ $\qquad \langle x, y\rangle = 0$

2) Consider $f(x) = a + x$, $g(x) = 2 - 3x^2$ in $L^2[0,1]$

$$\langle f, g\rangle = \int_0^1 (a+x)(2-3x^2)\,dx = a + \frac{1}{4}$$

hence orthogonal only if $a = -\frac{1}{4}$

3) An infinite set of mutually orthogonal functions

$$\{\phi_n(x)\} \equiv \{\sin(nx)\}_{n\geq 1} \subset L^2[0, 2\pi]$$

Easy to show

$$\langle \phi_j, \phi_k\rangle = 0 \iff i = j$$

Defn: Let $T = \{\phi_1, .. \phi_n\}$ be a basis of linear space S. We say $\alpha = (\alpha_1, .. \alpha_n)$ is the coordinate of $f \in S$ relative to T if

$$f = \sum_{i=1}^{n} \alpha_i \phi_i$$

Remark: Linear independence of ϕ_i assures the uniqueness of the coordinate. Why?

Coordinates for Inner Product Spaces S

EXAMPLE Let $T = \{\phi_1, .. \phi_n\}$ be a mutually orthogonal basis for S and

$$f = \sum_{i=1}^{n} \alpha_i \phi_i$$

Then

$$\langle f, \phi_k \rangle = \sum_{i=1}^{n} \alpha_i \langle \phi_i, \phi_k \rangle = \alpha_k \|\phi_k\|^2$$

so the coordinates are

$$\alpha_k = \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2}$$

Remark: Orthogonal basis $\Rightarrow$ easy to compute coordinates

EXAMPLE **Nonorthogonal basis**

Let $T = \{\phi_1, \ldots \phi_n\}$ be a basis for inner product space S.

(1) $$f = \sum_{i=1}^{n} \alpha_i \phi_i$$

(2) $$\langle f, \phi_j \rangle = \sum_{i=1}^{n} \alpha_i \langle \phi_i, \phi_j \rangle \qquad j = 1, \ldots n$$

Equation (2) is a matrix problem for unknown coodinates α_i.

(3) $$\Phi \vec{\alpha} = \vec{f}$$

where

$$\vec{\alpha} = (\alpha_1, \ldots \alpha_n) \qquad \Phi = [\langle \phi_i, \phi_j \rangle] \qquad \vec{f} = \begin{pmatrix} \langle f, \phi_1 \rangle \\ \vdots \\ \langle f, \phi_n \rangle \end{pmatrix}$$

The matrix Φ is invertible since if $f = 0$ in (1) and/or (3) we have the defn of independence of ϕ_i, i.e.,

$$\sum_{i=1}^{n} \alpha_i \phi_i = 0 \iff \alpha_i = 0 \quad \forall i$$

Since $\Phi \vec{\alpha} = \vec{0} \Rightarrow \vec{\alpha} = 0$ then Φ^{-1} exists and

(4) $$\vec{\alpha} = \Phi^{-1} \vec{f}$$

For orthogonal bases Φ is diagonal and easy to invert. Not so in (4).

Gram Schmidt orthogonalization

Let $T = \{x_1, \ldots x_n\}$ be a basis of inner production space S.

To find an orthogonal basis $T_\perp = \{\phi_1 \ldots \phi_n\}$

$$\phi_1 \equiv x_1$$

$$\phi_2 \equiv x_2 - \frac{\langle x_2, \phi_1 \rangle}{\|\phi_1\|^2} \phi_1$$

$$\phi_K \equiv x_K - \sum_{i=1}^{K-1} \frac{\langle x_k, \phi_i \rangle}{\|\phi_i\|^2} \phi_i$$

Pf/ Inductive (idea)

$$\langle \phi_2, \phi_1 \rangle = \langle x_2, \phi_1 \rangle - \frac{\langle x_2, \phi_1 \rangle}{\|\phi_1\|^2} \langle \phi_1, \phi_1 \rangle = 0$$

Thus $\phi_1 \perp \phi_2$. Next assume $\phi_K \perp \{\phi_1, \ldots \phi_{K-1}\}$ and show

$$\langle \phi_{K+1}, \phi_j \rangle = 0 \qquad \forall j = 1, \ldots k.$$

Example Legendre Polynomials

$$T = \{1, x, x^2, x^3\}$$

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx$$

Using Gram Schmidt can find orthogonal set:

$$T_\perp = \left\{ 1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5} x \right\}$$

remark on polynomial approximations.

Least Squares Approximation

Let $\dim(S) = n < \infty$ and $T = \{\phi_1, \ldots \phi_m\}$, $m \le n$ be an orthogonal linearly independent set. Suppose $f \in S$.

Not every $f \in S$ can be written $f = \sum^m \alpha_i \phi_i$ since $\dim(S) > m$. Still we may ask if $\exists \alpha_i$ such that the "distance"

$$(1) \qquad g(\alpha) \equiv \| f - \sum_{i=1}^{m} \alpha_i \phi_i \|^2$$

is smallest. We find $\bar{\alpha}$ s.t. $g(\bar{\alpha}) = \min_{\alpha} g(\alpha)$. Such an approximation is called the least square approximation of f relative to T. From (1)

$$\frac{\partial g}{\partial \alpha_j} = \frac{\partial}{\partial \alpha_j}\left(\|f\|^2 - 2\sum_{i=1}^{m} \alpha_i \langle f, \phi_i \rangle + \sum_{i=1}^{m} \alpha_i^2 \|\phi_i\|^2 \right)$$

$$\frac{\partial g}{\partial \alpha_j} = -2\langle f, \phi_j \rangle + 2\alpha_j \|\phi_j\|^2$$

Hence, necessarily,

$$\alpha_j = \frac{\langle f, \phi_j \rangle}{\|\phi_j\|^2} \qquad \forall j$$

This is the same as if T were a basis !!