# Series solutions involving SLP

(1) $$Lu = -(pu')' + qu \qquad B_K(u) = 0$$

for $x \in [a,b]$. Problem is self adjoint for separated or periodic B.C.

$$L\phi_n = \lambda w \phi_n \qquad B_K(\phi_n) = 0$$

are a complete set for $L^2_w[a,b]$. Assume $\|\phi_n\| = 1$

To solve

(2) $$Lu = f(x) \qquad u \in D(L)$$

we use expansions

$$u(x) = \sum_{n\geq 1} u_n \phi_n(x) \qquad f(x) = \sum_{n\geq 1} f_n \phi_n(x)$$

in eqn (2) and orthogonality under

$$\langle u, v\rangle_w \equiv \int_a^b u(x)v(x)w(x)\,dx$$

Explicitly

$$\sum_{n\geq 1} u_n L\phi_n = \sum_{n\geq 1} f_n \phi_n$$

So that

$$u_n = \frac{f_n}{\lambda_n} \qquad f_n = \langle f, \phi_n\rangle_w$$

yields a series soln for $u(x)$

$$u(x) = \sum_{n \geq 1} \frac{\langle f, \phi_n \rangle_w \, \phi_n(x)}{\lambda_n}$$

Alternately

$$u(x) = \int_a^b g(x,y)\, f(y)\, w(y)\, dy$$

where the <u>series representation</u> for $g(x,y)$ is

$$g(x,y) = \sum_{n \geq 1} \frac{\phi_n(x)\, \phi_n(y)}{\lambda_n}$$

This is the Green's function associated with $Lu = f$.

<u>EXAMPLE</u> $\quad Lu = u'' \quad , \quad u(0) = u(\pi) = 0 \quad \omega(x) \equiv 1$

Generates eigenfunctions $\phi_n(x) = \sin(nx)$. Solve

$$u'' = f(x) \qquad u(0) = u(\pi) = 0$$

$$\sum u_n \phi_n'' = \sum \frac{\langle f, \phi_n \rangle}{\| \phi_n \|^2} \phi_n$$

Hence

$$u_n = - \frac{\langle f, \phi_n \rangle}{n^2 \| \phi_n \|^2}$$

Were $f(x)$ it is readily verified

$$u(x) = 2 \sum_{n \geq 1} \frac{(-1)^n}{n^3} \sin(nx)$$

# Nonhomogeneous Boundary Conditions

$$(1) \qquad Lu = -(pu')' + qu = f(x)$$

$$(2) \qquad B_K(u) = \alpha_K u(x_K) + \beta_K u'(x_K) = \gamma_K \quad ; k=1,2$$

where $x_K = a, b$. Since $\gamma_K \neq 0$ the B.C. are said to be nonhomogeneous.

Goal is to find (first) any function $u_B(x)$ that satisfies the boundary conditions

$$B_K(u_B) = 0$$

If we then let

$$(3) \qquad u = u_H + u_B$$

it is easily verified that

$$(4) \qquad Lu_H = F(x) \equiv f(x) - Lu_B$$

$$(5) \qquad B_K(u_H) = 0$$

This latter problem has a Green's function $g(x,y)$ associated with it so that given (3)

$$u(x) = \int_a^b g(x,y) f(y)\, dy + u_B(x) - \int_a^b g(x,y) L u_B \, dy$$

The last two terms are associated with the inhomogoneity.

**EXAMPLE** **Nonhomogeneous B. Cond**

(1) $Lu \equiv u'' + u = f(x) \qquad x \in [0,\pi]$

(2) $u'(0) = 2$

(3) $u(\pi) = 0$

Choose a linear function $u_B = Ax + B$ to homogenize B.C.

$$u_B'(0) = A = 2$$

$$u_B(\pi) = A\pi + B = 0$$

Conclude

$$u_B(x) = 2(x - \pi)$$

Compute $Lu_B = 2(x-\pi)$ so that

$$u(x) = u_H(x) + u_B(x)$$

where $u_H(x)$ solves

(4) $u_H'' + u_H = f(x) - 2(x-\pi) = F(x)$

(5) $u_H'(0) = 0$

(6) $u_H(\pi) = 0$

Can solve this with associated e-fns:

$$L\phi_n = \lambda_n \phi_n$$

where (after some calculations) for $n \geq 0$

(7) $\phi_n(x) = \cos\beta_n x \quad , \beta_n = \frac{2n+1}{2}\pi \quad , \lambda_n = 1-\beta_n^2$

One can verify

$$\|\phi_n\|^2 = \int_0^\pi \cos^2\beta_n x \, dx = \frac{\pi}{2}$$

Seek a series soln to (4)-(6) using $\phi_n$.
Could solve using a Green's function too.

Since $\{\phi_n\}$ orthogonal

$$F(x) = \sum_{n\geq 0} F_n \phi_n \qquad F_n = \frac{\langle F, \phi_n\rangle}{\|\phi_n\|^2}$$

Of course $F_n$ depends on $f(x)$:

$$F_n = \frac{2}{\pi}\langle f, \phi_n\rangle - \frac{4}{\pi}\langle x-\pi, \phi_n\rangle$$

Some calculations yield

(8) $$F_n = \frac{2}{\pi}\langle f, \phi_n\rangle + \frac{4}{\pi\beta_n^2} \quad , n \geq 0$$

Now let

(9) $$u_H(x) = \sum_{n\geq 0} u_n \phi_n(x)$$

Using (8)-(9) in (4)

$$Lu_H = \sum_{n\geq 0} u_n\lambda_n\phi_n(x) = \sum_{n\geq 0} F_n \phi_n(x)$$

Orthogonality of $\phi_n$ then imply

$$u_n = \frac{1}{\lambda_n} F_n = \frac{2}{\pi} \frac{\langle f, \phi_n \rangle}{\lambda_n} + \frac{4}{\pi \lambda_n \beta_n^2}$$

where $\beta_n^2 = 1 - \lambda_n$.

## Summary

$$u(x) = 2(x-\pi) + \frac{2}{\pi} \sum_{n \geq 0} \frac{\langle f, \phi_n \rangle}{\lambda_n} \phi_n + \frac{4}{\pi} \sum \frac{\phi_n}{\lambda_n (1-\lambda_n)}$$

homogenizing function

soln if B.C. were homog.

extra term induced by inhomogeneous B.C.

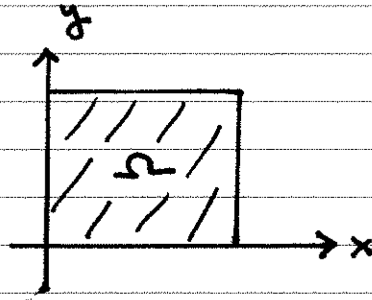
**Theorem** Let $\{\phi_n(x)\}$ and $\{\psi_n(y)\}$ be orthonormal bases for $L^2[a,b]$ and $L^2[c,d]$, respectively. Then $\Phi_{nm}(x,y) = \phi_n(x)\psi(y)$ are a ortho basis for $L^2(\Omega)$ where $\Omega = [a,b]\times[c,d]$.

This theorem is widely used in the method of separation of variables when solving linear partial differential equations.

**EXAMPLE** $\quad Lu \equiv \nabla^2 u = f(x,y)$ on $\Omega = [0,\pi]^2$

$$u|_{\partial\Omega} = 0$$

Seek eigenfunctions of $L$ of the form $\Phi = XY$. Here $X = X(x)$ and $Y = Y(y)$




$$\nabla^2\Phi = \lambda\Phi$$

(1) $$X''Y + XY'' = \lambda XY$$

Divide (1) by $XY$:

(2) $$\underset{\uparrow}{\frac{X''}{X}} + \underset{\uparrow}{\frac{Y''}{Y}} = \lambda$$

Indicated terms must be constant. Constants are the only functions of independent variables $x, y$ that can equal each other.

Given boundary conditions of $\Phi$ at $x = 0, \pi$ and $y = 0, \pi$ must have

(3) $$X'' + \alpha^2 X = 0 \qquad X(0) = X(\pi) = 0$$

(4) $$Y'' + \beta^2 Y = 0 \qquad Y(0) = Y(\pi) = 0$$

Here (3)-(4) are two SL-eigenvalue problems

$$X_m(x) = \sin(m\pi) \qquad \alpha_m = m = 1, 2, 3 \ldots$$

$$Y_n(x) = \sin(n\pi) \qquad \beta_n = n = 1, 2, 3 \ldots$$

Given (2) we conclude

$$\nabla^2 \Phi_{mn} = \lambda \Phi_{mn}$$

$$\Phi_{mn} = \sin(m\pi)\sin(n\pi)$$

$$\lambda_{mn} = -(m^2 + n^2)$$

These form an orthogonal basis for $L^2(\Omega)$

Now to solve the original problem

(5) $$u(x,y) = \sum_m \sum_n u_{mn} \Phi_{mn}(x,y)$$

(6) $$f(x,y) = \sum_m \sum_n f_{mn} \Phi_{mn}(x,y)$$

Inner product here is

$$\langle u, v \rangle = \int_0^\pi \int_0^\pi u v \, dy\, dx$$

Can verify

$$\| \Phi_{mn} \|^2 = \int_0^\pi \int_0^\pi \sin^2 mx \,|\, \sin^2 ny \, dy\, dx = \frac{\pi^2}{4}$$

Using (5)-(6) in $Lu = f$ and the orthogonalility of $\Phi_{mn}$

(7) $$\sum_m \sum_n \lambda_{mn} u_{mn} \Phi_{mn} = \sum_m \sum_n f_{nm} \Phi_{mn}$$

(8) $$u_{mn} = \frac{f_{mn}}{\lambda_{mn}} = \frac{\langle f, \Phi_{mn} \rangle}{\lambda_{mn} \| \Phi_{mn} \|^2}$$

Series solution for $u(x,y)$

$$u(x,y) = \sum_m \sum_n \frac{4}{\pi^2 \lambda_{mn}} \langle f, \Phi_{mn} \rangle \Phi_{mn}(x,y)$$

Becomes a Greens function

$$u(x,y) = \int_\Omega g(x,x',y,y') f(x',y') \, dx' dy'$$

where

$$g = \sum_m \sum_n \frac{4 \Phi_{mn}(x',y') \Phi_{mn}(x,y)}{\pi^2 \lambda_{mn}}$$