Sturm Liouville Self Adjoint operators

Consider the second order problem

$$(1) \qquad a_2 u'' + a_1 u' + a_0 u = f(x)$$

If $a_2(x) \neq 0$, this problem can be converted to a self adjoint form. Rewrite (1) as follows

$$(2) \qquad u'' + \frac{a_1}{a_2} u' + \frac{a_0}{a_2} u = \frac{1}{a_2} f(x)$$

Then multiply (2) by the integrating factor

$$p(x) = \exp\left(\int^x \frac{a_1(t)}{a_2(t)} dt \right) > 0$$

One then finds

$$\boxed{Lu \equiv -(pu')' + qu = \omega f}$$

where

$$q(x) = \frac{-a_0(x)}{a_2(x)} p(x)$$

$$\omega(x) = \frac{p(x)}{a_2(x)}$$

It is easily shown L is formally self adjoint

$$L\phi = L^*\phi \qquad \forall \phi \in D$$

Boundary Conditions

If one lets $\underline{u} = (u(a), u'(a), u(b), u'(b))^T$ then linear boundary conditions can be written

$$(3) \qquad B\underline{u} = 0$$

where

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix} \in \mathbb{R}^{2\times 4}$$

Equivalenty (3) may be written as a pair of boundary conditions

$$(4) \qquad B_k(u) = 0 \qquad k = 1, 2.$$

If each B_k only depends on evaluation at $x = a$ or $x = b$ exclusively then the boundary conditions are said to be <u>separated</u> in which case B has form:

$$B = \begin{bmatrix} x & x & 0 & 0 \\ 0 & 0 & x & x \end{bmatrix}$$

Otherwise they are <u>not separated</u> as is the case for <u>periodic</u> B.C

$$u(a) = u(b) \qquad u'(a) = u'(b)$$

in which case

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Regular Sturm Liouville Problem (SLP)

(i) $p(x) > 0$, $\omega(x) > 0$ $\forall x \in [a,b]$

(ii) $p, p', \omega, q \in C[a,b]$

(iii) a, b finite

Note closed interval is necessary. If operator L satisfies (i) - (iii) then it is a regular SLP.

If L is not regular it is singular

Special Boundary conditions

$u(a) = 0$ Dirichlet B.C. at $x=a$

$u(b) = 0$ Dirichlet B.C. at $x=b$

$u'(a) = 0$ Neumann B.C. at $x=a$

$u'(b) = 0$ Neumann B.C. at $x=b$

Otherwise

$$B_k(u) = \alpha_{1k} u(x_j) + \alpha_{2k} u'(x_j) = 0$$

is a mixed or Robin B.C. at $x = x_j$.

Self Adjointness

Using the $L^2[a,b]$ inner product

(1) $\langle Lu, v\rangle = J(u,v) + \langle u, Lv\rangle$

where the boundary term is

(2) $J = p(uv' - u'v)\Big|_a^b$

For $B_1(u) = \alpha_1 u(a) + \alpha_2 u'(a) = 0$ one can show

$$p(uv' - u'v)\Big|_{x=a} = \frac{1}{\beta} p(a) u(a) B_1(v)$$

Thus $B_1(u) = 0 \iff B_1(v) = 0$. Thus Dirichlet, Neumann and Mixed B.C result in self adjoint B.C. Same is true (from (2)) that if $u \in D(L)$ hap periodic B.C. so does $D(L^*)$.

Summary

If a regular SLP has mixed (separated) B.C. or periodic B.C. the operator is self adjoint

$$L = L^*$$

$$D(L) = D(L^*)$$

Green's Function for Regular SLP

$$Lu = -(pu')' + qu = f(x) \qquad x \in [a,b]$$

$$D(L) = \{ u \in C^2[a,b] : \; B_1(u) = B_2(u) = 0 \}$$

has solution

$$u(t) = \int_a^b g(x,t)\, f(x)\, dx$$

where

$$(1) \quad g(x,t) = \begin{cases} g_+(x,t) \equiv \dfrac{-u_1(x)\,u_2(t)}{p(t)\,W(t)} & x < t < b \\[2ex] g_-(x,t) \equiv \dfrac{-u_1(t)\,u_2(x)}{p(t)\,W(t)} & a < t < x \end{cases}$$

where $u_k(x)$ are independent solns of

$$(2) \quad Lu_1 = 0 \qquad B_1(u_1) = 0 \qquad x = a$$

$$(3) \quad Lu_2 = 0 \qquad B_2(u_2) = 0 \qquad x = b$$

and W is the Wronskian

$$W(x) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2$$

Proof is to verify same conditions that apply to L in general form.

EXAMPLE (Using SLP general result)

(1) $\quad Lu = -(xu')' \qquad u(1) = 0 \qquad u'(2) = 0$

is a self adjoint problem. Must solve

(2) $\quad -(xu_1')' = 0 \qquad u_1(1) = 0$

(3) $\quad -(xu_2')' = 0 \qquad u_2'(2) = 0$

General soln of $Lu = 0$ is $u(x) = c_1 + c_2 \ln x$.
Using this we find

$$u_1(x) = \ln x \qquad u_2(x) = 1$$

Calculate Wronskian

$$W = u_1 u_2' - u_1' u_2 = -\frac{1}{x}$$

Using general result for 2nd order BVP, $pW = -1$,

$$g(x,t) = \begin{cases} \ln x & x < t < 2 \\ \ln t & 1 < t < x \end{cases}$$

Eigenfunctions of Regular SLP

Let $Lu = -(pu')' + qu$ have a domain defined by separated or periodic boundary conditions and define

$$(1) \qquad \langle u, v \rangle_w = \int_a^b w(x)\, u(x)\, v(x)\, dx$$

where $w(x) > 0$ assumed. Let ϕ_i be eigenfunctions of

$$-(p\phi_i')' + q\phi_i = \lambda w \phi_i$$

then

(i) $\lambda \in \mathbb{R}$

(ii) $\lambda_i \neq \lambda_j \Rightarrow \langle \phi_i, \phi_j \rangle_w = 0$

(iii) $\{\phi_i\}$ are a complete set for $L^2[a,b]$ and orthogonal under the weighted inner product (1).

<u>Note</u>: existence of a complete set is a consequence of existence of a symmetric Green's function

Proof (ii) Let $\lambda_i \neq \lambda_j$

(1) $\quad -(p\phi_i')' + (q - \lambda_i w)\phi_i = 0$

(2) $\quad -(p\phi_j')' + (q - \lambda_j w)\phi_j = 0$

Perform $\phi_j(1) - \phi_i(2)$ to find

$$(\lambda_j - \lambda_i) w \phi_i \phi_j = \phi_j (p\phi_i')' - \phi_i (p\phi_j')'$$

$$= \frac{d}{dx}\left\{ (p\phi_i')\phi_j - (p\phi_j')\phi_i \right\}$$

Integrate over $[a,b]$

$$(\lambda_j - \lambda_i) \langle \phi_i, \phi_j \rangle_w = J(\phi_i, \phi_j) \Big|_a^b$$

Given $\phi_i, \phi_j \in D(L)$ we have

$$(\lambda_j - \lambda_i) \langle \phi_i \phi_j \rangle_w = 0$$

$$\langle \phi_i, \phi_j \rangle_w = 0 \qquad \square$$

Regular SL Eigenvalue Problem

(1) $$Lu = \lambda \omega u \qquad u \in D(L)$$

weighted $L^2_\omega[a,b]$ inner product and

$$Lu \equiv -(pu')' + qu \qquad x \in [a,b]$$

Suppose the domain is defined by general mixed B.C.

(2) $$B_K(u) = \alpha_{1K} u + \alpha_{2K} u' = 0 \qquad x = a, b$$

Let $u_1(x)$ and $u_2(x)$ be two independent solutions of

$$Lu = \lambda \omega u$$

The general solution is

(3) $$u(x;\lambda) = c_1 u_1(x) + c_2 u_2(x)$$

where u_K also depend on λ.

We seek a nontrivial solution (3) which satisfies (2). Such a solution is an eigenfunction.

Since the boundary operators are linear:

$$B_K(c_1 u_1 + c_2 u_2) = c_1 B_K(u_1) + c_2 B_K(u_2)$$

Thus the requirement both B.C. be satisfied results in a linear system for the constants c_1 and c_2:

$$c_1 B_1(u_1) + c_2 B_1(u_2) = 0$$

$$c_2 B_2(u_1) + c_2 B_2(u_2) = 0$$

which can be written

$$(4) \qquad A(\lambda)\vec{c} = \vec{0} \qquad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

for

$$A(\lambda) = \begin{bmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{bmatrix}$$

Thus an eigenvalue of L is a root of

$$\det A(\lambda) = 0$$

For such λ, $N(A) \neq 0$ so $\exists\, \vec{c} \neq \vec{0}$ solving (4).

(i) $\det A(\lambda_n) = 0$

(ii) $\phi_n(x) = c_1 u_1(x) + c_2 u_2(x) \qquad \vec{c} \in N(A)$

(iii) $\langle \phi_i, \phi_j \rangle_w = 0 \qquad i \neq j$

Normalization w.r.t. L^2_w inner product.

Dirichlet and Neumann B.C.

Let ϕ be an eigenfunction satisfying

$$(1) \qquad L\phi = \lambda w \phi \qquad \phi \in D(L)$$

Using the standard $L^2[a,b]$ inner product

$$\langle L\phi, \phi \rangle = \int_a^b \left(-(p\phi')' + q\phi\right)\phi \, dx$$

$$\langle L\phi, \phi \rangle = -p\phi\phi'\Big|_a^b + \int_a^b \left(p(\phi')^2 + q\phi^2\right) dx$$

Using (1) we find

$$(2) \qquad \lambda \langle w\phi, \phi \rangle = -p\phi\phi'\Big|_a^b + \int_a^b \left(p(\phi')^2 + q\phi^2\right) dx$$

<u>Theorem</u> If the SLP (1) has only Dirichlet or Neumann B.C and $q(x) \geq 0$ on $[a,b]$ then

$$\lambda \geq 0$$

<u>Proof</u> The boundary term in (2) vanishes. If $\phi' \neq 0$ the right side is strictly positive since $p(x) > 0$, $q(x) \geq 0$. Hence $\lambda > 0$. If $q(x) = 0$ and $\phi' = 0$ then (2) implies $\lambda \langle w\phi, \phi \rangle = 0 \Rightarrow \lambda = 0$. This case occurs if $\phi =$ constant for pure Neumann B.C. $\square$

EX Find all the eigenfunctions for SLP

$$u'' + \lambda u = 0 \qquad u(0) = u(\pi) = 0$$

Here $p(x) = -1$, $q(x) = 0$, $\omega(x) = 1$.

Since B.C. are Dirichlet $\lambda \geq 0$

CASE $\lambda = 0$

$$u(x) = c_1 x + c_2$$

Boundary conditions imply $c_1 = c_2 = 0$ so $\lambda = 0$ is not an eigenvalue.

CASE $\lambda > 0$ $\qquad \lambda = \mu^2$

$$u(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x)$$

Boundary conditions result in system

$$\begin{aligned} u(0) &= & & c_2 &= 0 \\ u(\pi) &= \sin\mu\pi \; c_1 &+ \cos\mu\pi & \; c_2 &= 0 \end{aligned}$$

Nontrivial solutions only if

$$\det A = -\sin\mu\pi = 0$$

Conclude

$$\lambda_n = n^2 \qquad n = 1, 2, \ldots$$

$$\phi_n(x) = \sin(nx)$$

EX Find all the eigenfunctions for SLP

$$u'' + \lambda u = 0 \qquad u'(0) = u'(\pi) = 0$$

We know $\lambda \geq 0$.

CASE $\lambda = 0$

$$\phi_0(x) = 1 \qquad \lambda_0 = 0$$

satisfies ODE and B.C. for $\lambda = 0$.

CASE $\lambda = \mu^2 > 0$

$$u(x) = c_1 \sin(\mu x) + c_2 \cos \mu x$$

The boundary conditions yield

$$A = \begin{bmatrix} \mu & 0 \\ \mu\cos(\mu\pi) & -\mu\sin(\mu\pi) \end{bmatrix}$$

$$\det A = -\mu^2 \sin(\mu\pi)$$

Also clear $c_1 = 0$ so we conclude

$$\lambda_n = n^2 \qquad \phi_n(x) = \cos(nx) \qquad n \geq 1$$

Collectively we may expand $f(x) \in L^2[a,b]$:

$$f(x) = f_0 \phi_0(x) + \sum_{n \geq 1}^{\infty} f_n \phi_n(x)$$

using orthogonality under $\langle u, v \rangle_w$ weighted inner product. Here just regular L^2.

EX Eigenfunctions of

(1) $u'' + \lambda u = 0 \qquad u(0) - u'(0) = u(\pi) - u'(\pi) = 0$

This problem has a negative eigenvalue!

CASE $\lambda = -\mu^2 < 0$

$$u(x) = c_1 \cosh \mu x + c_2 \sinh \mu x$$

After some lengthy calculations the boundary conditions imply $A\vec{c} = 0$

(2) $$A = \begin{bmatrix} 1 & \mu \\ \cosh \mu\pi - \mu \sinh \mu\pi & \mu \cosh \mu\pi - \sinh \mu\pi \end{bmatrix}$$

For such λ a nontrivial solution exists only if

$$\det A = (\mu^2 - 1) \sinh \mu\pi = 0$$

only if $\mu^2 = 1$ so that $\lambda = -1$ eigenvalue. For $\mu = -1$ in (2) we see $c_1 = c_2$ yields

$$u(x) = c_1 (\cosh \mu x + \sinh \mu x)$$

This simplifies using defn of cosh, sinh.

Conclude:

$$\lambda = -1 \qquad u(x) = e^x$$

CASE $\lambda = 0$

One can show $u(x) = c_1 + c_2 x$ and boundary conditions imply $c_1 = c_2 = 0$ so $u(x) \equiv 0$ and $\lambda = 0$ not an eigenvalue.

CASE $\lambda = +\mu^2 > 0$

$$u(x) = c_1 \cos \mu x + c_2 \sin \mu x$$

Proceed in a similar fashion to conclude $\vec{c} = (c_1, c_2)$ is a solution of $A\vec{c} = \vec{0}$ where

$$(3) \qquad A = \begin{bmatrix} 1 & -\mu \\ \cos\mu\pi + \mu \sin\mu\pi & \sin\mu\pi - \mu\cos\mu\pi \end{bmatrix}$$

Eigenvalue equation

$$\det A = (1+\mu^2)\sin\mu\pi = 0$$

Hence $\mu = n$ and $\lambda_n = n^2$ are eigenvalues. For these λ the first row of (3) implies

$$c_1 - n c_2 = 0$$

and we conclude the eigenvalue/function pairs:

$$(4) \qquad \lambda_n = n^2 \qquad u_n(x) = (n\cos(nx) + \sin nx)$$

<u>Conclusion</u> The eigenfunctions for $\lambda < 0$, $\lambda > 0$ cases collectively form an orthogonal basis for $L^2[0,\pi]$

$$f(x) = \underbrace{f_0 e^x}_{\lambda<0} + \sum_{n=1}^{\infty} \underbrace{f_n (n\cos nx + \sin nx)}_{\lambda > 0}$$

Since $w(x) = 1$ orthogonality of $\{u_n\}_{n\geq 0}$ is under standard $L^2[0,\pi]$ inner product.

EX

$$\boxed{\begin{array}{lll} -(xu')' - \frac{2}{x} u = \frac{\lambda}{x} u & & x \in [1,2] \\ u'(1) = 0 & u'(2) = 0 & \end{array}}$$

Here we have $p(x) = x$, $q(x) = -\frac{2}{x}$, $\omega(x) = \frac{1}{x}$, and Neumann B.C.

Rearranging we obtain Cauchy-Euler Eqn

$$x^2 u'' + x u' + (\lambda + 2) u = 0$$

Characteristic equation found by letting $u = x^r$

$$r^2 + (\lambda + 2) = 0$$

For $(\lambda + 2) > 0$ we get the general soln

$$u(x) = c_1 \cos(\sqrt{\lambda + 2} \ln x) + c_2 \sin(\sqrt{\lambda + 2} \ln x)$$

Using boundary conditions we ultimately arrive at

$$\lambda_n = -2 + \left(\frac{n\pi}{\ln 2}\right)^2 \qquad u_n = \cos\left(\frac{n\pi}{\ln 2} x\right)$$

for $n = 1, 2, 3, \ldots$

For $(\lambda + 2) = 0$ we have $r^2 = 0$ and

$$u(x) = c_1 + c_2 \ln x$$

Boundary conditions imply $c_2 = 0$ and

$$\lambda_0 = -2 \qquad u_0 \equiv 1$$