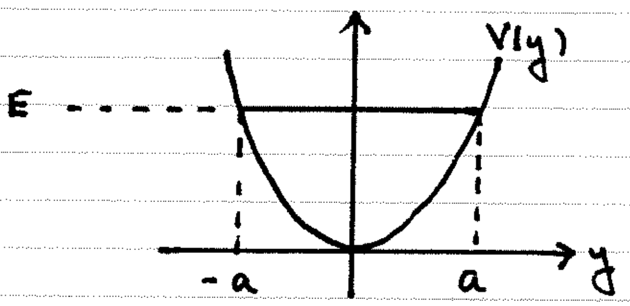
Adiabatic Invariants

In perturbed Hamiltonian systems are functions $A(q,p,t)$ that remain constant over long periods of time.

EX 1 $\quad y'' + \omega_0^2 y = 0$

$$E \equiv \tfrac{1}{2}(y')^2 + V(y) \qquad \text{energy}$$

$$V = \tfrac{1}{2}\omega_0^2 y^2 \qquad \text{potential energy}$$



Solution $y(t) \in [-a, a]$

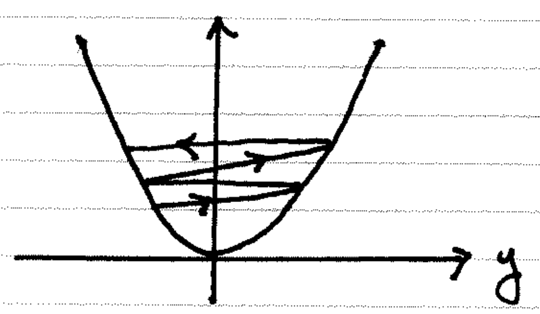
$$\frac{dE}{dt} = 0$$

hence E invariant

EX 2 $\quad y'' + \omega(\tilde{t})^2 y = 0 \qquad \tilde{t} = \varepsilon t$ slow

$$E \equiv \tfrac{1}{2}(y')^2 + \tfrac{1}{2}\omega^2 y^2 \qquad \text{energy}$$

$$\frac{dE}{dt} = \varepsilon\, \omega \frac{d\omega}{d\tilde{t}}\, y^2 \qquad \text{not invariant}$$



Slow drift up if $\omega' > 0$

Asymptotic growth of "invariant"

$$y'' + \omega(\tilde{t})^2 y = 0$$

leading multiple scales solution

$$(1) \qquad Y_0(t, \tilde{t}) = A_0(\tilde{t}) \cos(\psi) \qquad \psi = t - \phi_0(\tilde{t})$$

where

$$A_0(\tilde{t}) = A_0(0) \sqrt{\frac{\omega(0)}{\omega(\tilde{t})}} \qquad \phi_0(\tilde{t}) = \phi_0(0)$$

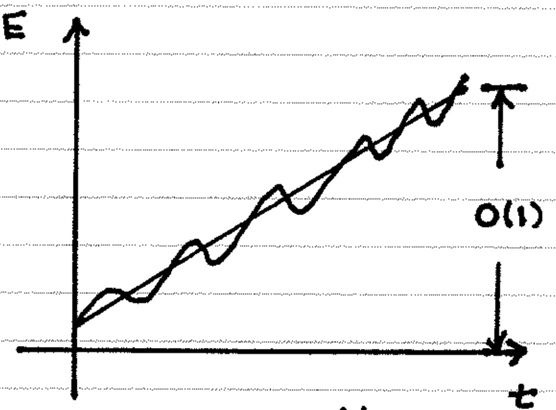
Examine growth in E (leading invariant)

$$\frac{dE}{dt} = \varepsilon \omega \omega' y^2$$

$$\frac{dE}{dt} = \varepsilon \omega \omega' Y_0^2 + O(\varepsilon^2)$$

(1)

$$\frac{dE}{dt} = \frac{1}{2} \varepsilon \omega' A_0(0)^2 \omega(0) \Big\{ 1 + \underbrace{2\cos 2\psi}_{\text{zero average}} \Big\} + O(\varepsilon^2)$$



$$t = O\left(\frac{1}{\varepsilon}\right)$$

Here average is

$$\langle E \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} E(\psi, \tilde{t})\, d\psi$$

Adiabatic invariant

Recast problem as perturbed or slowly varying Hamiltonian

(1) $$H(p, q) = \frac{1}{2} p^2 + \frac{1}{2} \omega(\hat{t})^2 q^2$$

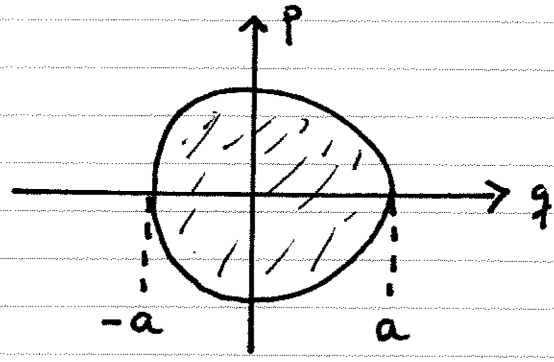
Equations of motion

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Hamilton's equations

Were $\omega(\hat{t})^2$ constant, trajectories are



From I. Cond.

$$a = \sqrt{\frac{2E}{\omega(0)^2}}$$

Define the <u>action</u> J by

$$\boxed{J \equiv \oint p\, dq}$$

In this case the action is the area enclosed.

For the specific Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 = E$$

one solves for p in terms of q to get

$$J = \oint \sqrt{2E - \omega^2 q^2}\, dq$$

$$J = \frac{1}{\omega} \oint \sqrt{a^2 - q^2}\, dq \qquad a^2 = \frac{2E}{\omega^2}$$

$$J = \frac{2}{\omega} \int_{-a}^{a} \sqrt{a^2 - q^2}\, dq$$

$$J = \omega \pi a^2$$

$$\boxed{J = \frac{2\pi E}{\omega}}$$

__Growth of J__

$$\frac{dJ}{dt} = -\frac{2\pi \varepsilon \omega' E}{\omega^2} + \frac{2\pi}{\omega}\frac{dE}{dt}$$

defn of E

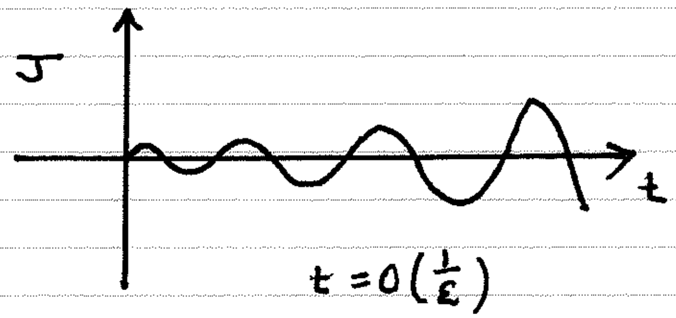
$$\frac{dJ}{dt} = \varepsilon \pi \omega' \left(y^2 - \frac{1}{\omega^2}(y')^2 \right)$$

Using $y \sim \Phi_0(t,\tilde{t}) = A_0(\tilde{t}) \cos\psi$ we find

$$\frac{dJ}{dt} = \underline{\varepsilon} \pi \omega' A_0^2 \cos(2\psi) + O(\varepsilon^2)$$

has zero average to $O(\varepsilon)$.

$$\boxed{J = \frac{\pi}{\omega}(y')^2 + \pi \omega y^2}$$



a "better" invariant

<u>Definition</u> An adiabatic invariant of (to $O(\varepsilon^n)$)

$$f(y'', y', t; \varepsilon) = 0$$

is a function $A(y', y, t, \varepsilon)$ s.t.

$$\frac{dA}{dt} = \varepsilon^{n+1} \phi_{n+1}(y, y', t; \varepsilon) \qquad \tilde{t} = O(1)$$

with

$$\langle \phi_{n+1} \rangle = 0$$

in some phase ψ.