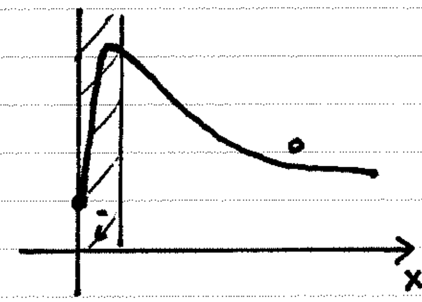
Overview of Linear BVP

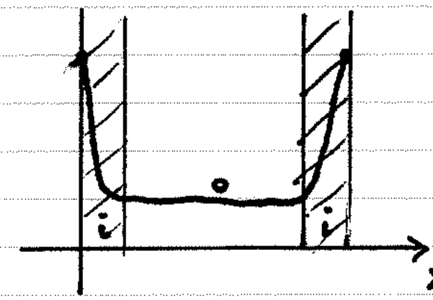
$$(1) \qquad \varepsilon y'' + a(x,\varepsilon)y' + b(x,\varepsilon)y = f(x,\varepsilon)$$

$$(2) \qquad y(0,\varepsilon) = A \qquad y(1,\varepsilon) = B$$

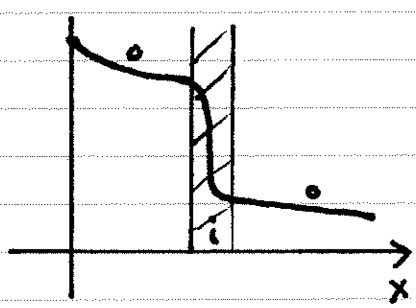
where $\varepsilon \ll 1$. In general $y(x,\varepsilon)$ exhibits singular behavior and has layers



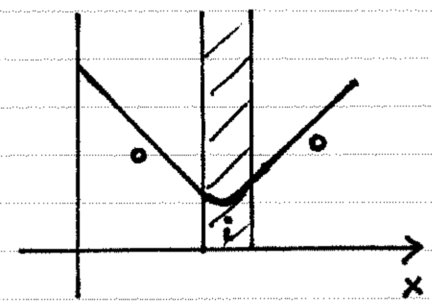
single layer



two layers



interior layer



corner layer

In the layers $\varepsilon y''$ is not small.

i inner layer

o outer layer

Existence of unique solutions

Letting $D = \{ y \in C^2[0,1] : y(0) = A,\ y(1) = B \}$ the problem (1)-(2) may compactly be written

$$(3) \qquad L(y) = f \qquad y \in D$$

A necessary condition that (3) have a unique soln is that the only solution to

$$(4) \qquad L(y) = 0 \qquad y(0) = y(1) = 0$$

is $y(x) \equiv 0$. To simplify (4) we use an integrating factor and rewrite

$$y(x) = \exp\left(-\frac{1}{2\varepsilon} \int_0^x a(s)\,ds \right) w(x)$$

Then $w(x)$ is a soln to

$$(5) \qquad \varepsilon w'' - p(x) w = 0$$

$$(6) \qquad w(0) = w(1) = 0$$

where

$$(7) \qquad p(x) = \frac{a^2}{4\varepsilon} + \frac{a'}{2} - b$$

Then $y(x) \equiv 0$ is the only solution of (4) iff $w(x) = 0$ is the only soln of (5)-(6)

From (5) we note

$$\varepsilon w w'' - p w^2 = 0$$

Hence

$$\varepsilon \int_0^1 w w'' dx = \int_0^1 p w^2 dx$$

Integrating by parts and using $w(0) = w(1) = 0$ one gets

$$(8) \qquad \boxed{-\varepsilon \int_0^1 (w')^2 dx = \int_0^1 p(x) w^2 dx}$$

Notice that if $p(x) < 0$ on some subinterval of $[0,1]$ this identity could be true for some $w \neq 0$. However, if $p(x) > 0$ on $[0,1]$ the only soln of (8) is $w = 0$ making the soln of (3) unique. Suff. cond. for uniqueness

$$p(x) = \frac{a^2}{4\varepsilon} + \frac{a'}{2} - b > 0$$

So long as a, a' and b are continuous on $[0,1]$

$$a(x) \neq 0 \qquad \forall x \in [0,1]$$

assures unique solns as $\varepsilon \to 0^+$.

Single boundary layer example

(1) $\qquad \varepsilon y'' + a(x) y' + b(x) y = 0 \qquad x \in (0,1)$

(2) $\qquad y(0) = A \qquad y(1) = B$

(3) $\qquad a(x) > 0 \quad \text{on } [0,1]$

We also assume $a, b \in C^1[0,1]$.

Outer problem

$$y(x,\varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$$

We assume $y_0(x)$ satisfies the right B.C.

$$a(x) y_0' + b(x) y_0 = 0$$

has the solution $\qquad a \neq 0$

$$y_0(x) = B \exp\left(\int_x^1 \frac{b(s)}{a(s)} \, ds \right)$$

Inner problem

$$y(x,\varepsilon) = Y(X, \varepsilon)$$

$$X = \frac{x}{\varepsilon}$$

Here there is a boundary layer at $x=0$ of thickness $\delta = O(\varepsilon)$

Differential equation

(4) $$Y'' + a(\varepsilon X)Y' + \varepsilon b(\varepsilon X)Y = 0$$

(5) $$Y(0,\varepsilon) = A$$

Using

$$Y(X,\varepsilon) = Y_0(X) + \varepsilon Y_1(X) + O(\varepsilon^2)$$

the leading problem is

$$Y_0'' + a(0)Y_0' = 0 \quad , \quad Y_0(0) = A$$

whose solution is

(6) $$Y_0(X) = A + C(1 - e^{-a(0)X})$$

where $C \in \mathbb{R}$ is a constant to be found using matching

Matching

Since $a(0) > 0$ the inner solution Y_0 is bounded as $X \to \infty$. Consequently the inner and outer solns can be matched:

(7) $$\lim_{x\to 0^+} y_0(x) = \lim_{X\to\infty} Y_0(X)$$

Were $a(0) < 0$ the layer must be at $x = 1$ since otherwise $Y_0(X) \to \pm\infty$ can't be matched.

Prandtl matching (7) yields

$$M_0 = \lim_{x\to 0^+} y_0(x) = B\exp\left(\int_0^1 \frac{b(s)}{a(s)}\,ds\right) = A + C = \lim_{X\to\infty} Y_0(X)$$

This detemines C and the <u>composite soln</u>

part that matches in overlap domain.

(8) $$y_c(x,\varepsilon) \equiv y_0(x) + Y_0(X) - M_0$$

or

$$y_c(x,\varepsilon) = B\exp\left(\int_x^1 \frac{b(s)}{a(s)}\,ds\right) + (A - M_0)\,e^{-a(0)x/\varepsilon}$$

Note $y_c(1,\varepsilon) = B$ whereas $y_c(0,\varepsilon) = A + O(\varepsilon)$.
Also

$y_c \sim y_0(x)$ x fixed

$y_c \sim Y_0(X)$ X fixed

<u>Remarks</u>

$a(x) > 0$	$x \in [0,1]$	Layer at $x=0$
$a(x) < 0$	$x \in [0,1]$	Layer at $x=1$
$a(\bar{x}) = 0$	$\bar{x} \in (0,1)$	Interior Layer
$a(x) = 0$	$x = 0$	Corner Layer Depends.

Example Two boundary layers

(1) $\varepsilon y'' - y = -1 \qquad y(0)=0 \quad y(1)=2$

Outer problem

$$y_0(x) \equiv 1$$

can't match either boundary condition hence two boundary layers.

Inner problems

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x-\bar{x}}{\varepsilon^{\beta}} \qquad \bar{x} = 0, 1$$

Substitute into (1) yields

$$\varepsilon^{1-2\beta} Y'' - Y = -1$$

Dominant balance $\Rightarrow \beta = \frac{1}{2}$.

Note that $X \to \infty$ for $\bar{x} = 0$ and $X \to -\infty$ for $x = 1$ (outer limits)

$$\bar{x} = 1 \qquad Y_0^+(X) = c_1^+ e^{X} + (1-c_1^+) e^{-X} + 1$$

$$\bar{x} = 0 \qquad Y_0^-(X) = -(1+c_2^-) e^{X} + c_2^- e^{-X} + 1$$

These satisfy the Bound. Conditions

$$Y_0^-(0) = 0 \qquad Y_0^+(0) = 2$$

Matching (Prandtl)

The outer solution can match these inner expansions only if:

$$M^+ = \lim_{x \to 1^-} y_0(x) = \lim_{X \to -\infty} Y_0^+(X)$$

$$M^- = \lim_{x \to 0^+} y_0(x) = \lim_{X \to \infty} Y_0^-(X)$$

Considering the expressions for $Y_0^{\pm}(X)$ these limits exist, are finite and equal only if

$$c_1^+ = 1 \qquad c_2^- = -1$$

Hence

$$Y_0^+(X) = 1 + e^{X} \qquad X = \frac{x-1}{\varepsilon^{1/2}}$$

$$Y_0^-(X) = 1 - e^{-X} \qquad X = \frac{x}{\varepsilon^{1/2}}$$

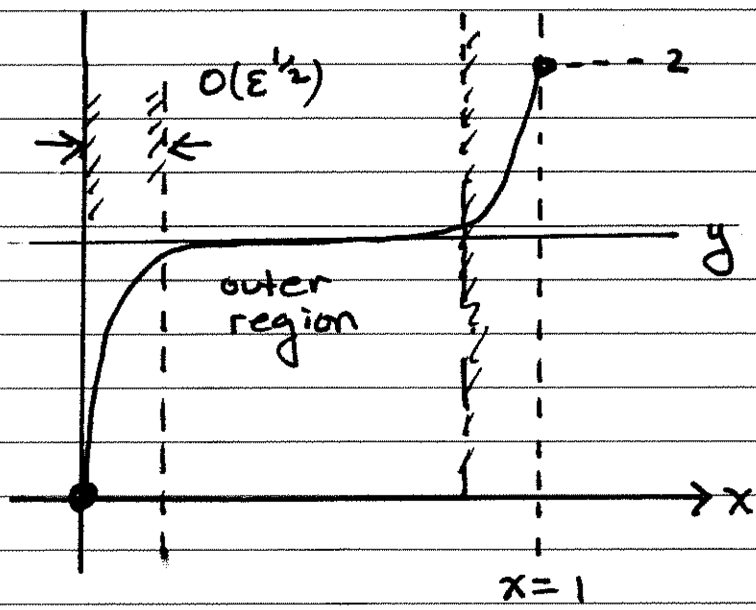
Regardless $M^{\pm} = 1$, and composite solution

$$y_c(x,\varepsilon) = y_0(x) + Y_0^+(X) + Y_0^-(X) - M^+ - M^-$$

Thus

$$y_c(x,\varepsilon) = 1 + \exp\left(\frac{x-1}{\varepsilon^{1/2}}\right) - \exp\left(\frac{x}{\varepsilon^{1/2}}\right)$$

Graph of asymptotic approximation



$$y_c(x,\varepsilon) \sim y_0(x) = 1 \quad \text{in outer region.}$$

EXAMPE Interior Layer

$$\boxed{\begin{array}{ll} \varepsilon y'' + x y' + 3x^3 y = 0 & x \in (-1,1) \\ y(-1) = 4e \qquad y(1) = 2e^{-1} & \end{array}}$$

Since $a(x) = x$ vanishes at $\bar{x} = 0$ one might expect an interior layer at $\bar{x} \in (-1,1)$.

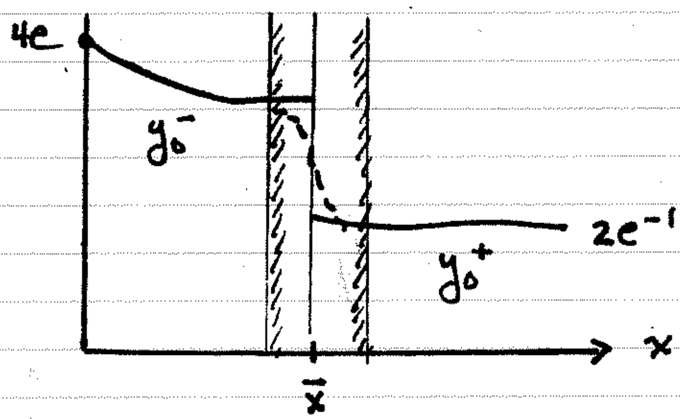
Outer problems

$$x y_0' + 3x^3 y_0 = 0$$

Seek two outer solns in the two outer regions $x > 0$ and $x < 0$.

$$x < 0 \qquad y_0^-(x) = 4e^{-x^3} \qquad y_0^-(-1) = 4e$$

$$x > 0 \qquad y_0^+(x) = 2e^{-x^3} \qquad y_0^+(1) = 2e^{-1}$$



Need inner soln --- to connect outer solns in layer

$$\lim_{x \to \bar{x}} y_0^-(x) = 4 \neq 2 = \lim_{x \to \bar{x}} y_0^+(x)$$

Inner Problem

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x-\bar{x}}{\delta(\varepsilon)} \qquad 0<\delta\ll 1$$

Here the (interior) layer location is $\bar{x}=0$ and $\delta(\varepsilon)$ is its thickness to be determined.

$$Y'' + \frac{\delta^2}{\varepsilon} X Y' + \frac{\delta^5}{\varepsilon}(3X^3Y) = 0$$

$$① \sim ② \gg ③$$

Only possible dominant balance (that leads to a Y_0 which can be matched to both outer solutions)

$$\delta(\varepsilon) = \varepsilon^{1/2}$$

Leading problem

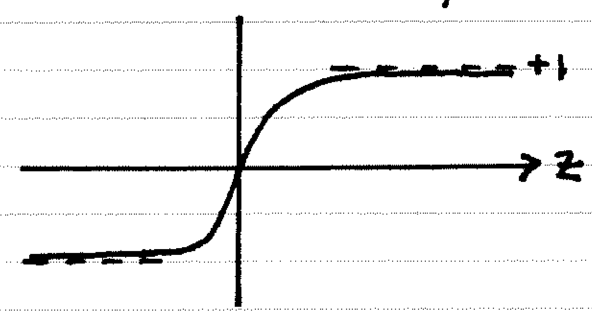
$$Y_0'' + X Y_0' = 0$$

has the general solution

$$Y_0(X) = c_1 + c_2 \operatorname{erf}\left(\frac{X}{\sqrt{2}}\right)$$

where the error function is defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

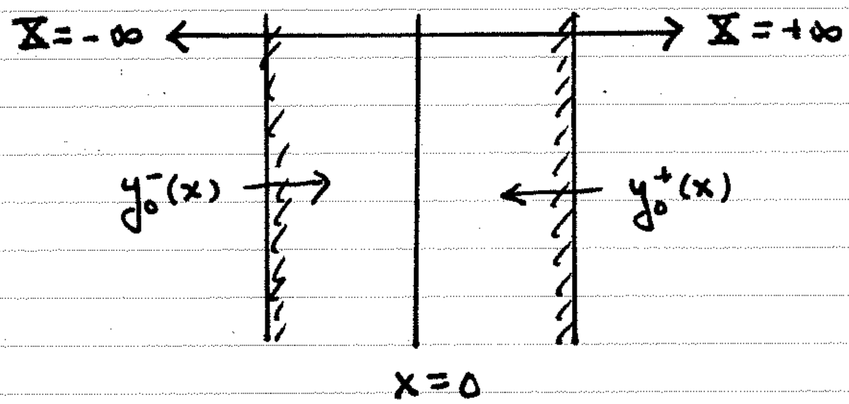


Matching conditions

$$M_+ = \lim_{x \to 0^+} y_0^+(x) = \lim_{X \to \infty} Y_0(X)$$

$$M_- = \lim_{x \to 0^-} y_0^-(x) = \lim_{X \to -\infty} Y_0(X)$$

Each outer solution must match the (sole) inner solution in their respective overlap domains.



Matching conditions

$$M_+ = 2 = c_1 + c_2$$

$$M_- = 4 = c_1 - c_2$$

$c_1 = 3$, $c_2 = -1$

Hence

$$Y_0(X) = 3 - \mathrm{erf}\left(\frac{X}{\sqrt{2}}\right)$$

Composite solution (piecewise defined)

$$y_c^{\pm}(x, \varepsilon) = y_0^{\pm}(x) + Y_0\left(\frac{x}{\sqrt{2\varepsilon}}\right) - M_{\pm}$$

More explicitly

$$y_c(x, \varepsilon) = \begin{cases} 4e^{-x^3} - \operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) - 1 & x < 0 \\ 2e^{-x^3} - \operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) + 1 & x > 0 \end{cases}$$

For this approximation one can verify

$$\left.\frac{\partial^k y_c}{\partial x^k}\right|_{x=0^-} = \left.\frac{\partial^k y_c}{\partial x^k}\right|_{x=0^+}$$

for $k = 0, 1, 2$ but not for $k \geq 3$. Despite the piecewise defn $y_c \in C^2[-1, 1]$

EXAMPLE Varied interior layer thickness

$$\varepsilon y'' + x^p y' = 0 \qquad y(-1) = 1, \; y(1) = 3$$

Here $a(x) = x^p$ vanishes at a turning point $\bar{x} = 0$

Outer solutions

Regardless of the region of validity

$$x^p y_0' = 0$$

$$y_0 = c$$

At $x = 0$, $\varepsilon y'' = O(x^p y')$ so expect a layer possible.

Inner problem

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x - \bar{x}}{\delta(\varepsilon)} \qquad \bar{x} = 0$$

yields

$$\underset{①}{Y''} + \underset{②}{\frac{\delta^{p+1}}{\varepsilon} X^p Y'} = 0$$

$$① \sim ②$$

Balance determines layer thickness

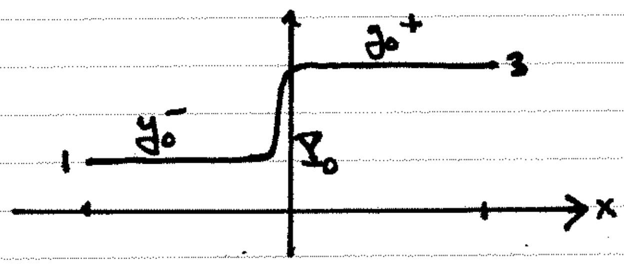
$$\delta(\varepsilon) = \varepsilon^{\frac{1}{p+1}}$$

Remark: The case ① ≫ ② yields $Y_0(X) = AX + B$ which can't be matched to any outer solution.

	p	$\delta(\varepsilon)$	$Y_0(X)$	Shape
BL	0	ε	$A + Be^{-X}$	
IL	1	$\varepsilon^{1/2}$	$A + B\,\mathrm{erf}\left(\frac{X}{\sqrt{2}}\right)$	
BL	2	$\varepsilon^{1/3}$	$A + B\int_0^X \exp(-\frac{1}{3}s^3)\,ds$	
IL	3	$\varepsilon^{1/4}$	$A + B\int_0^X \exp(-\frac{1}{4}s^4)\,ds$	

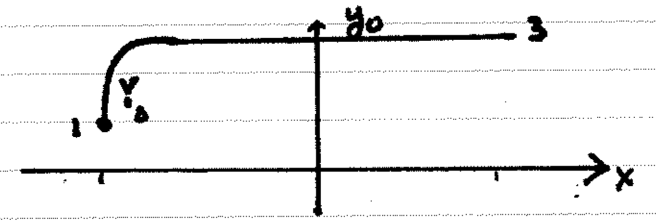
Shapes determine if an interior layer is possible ($p=1,3$). For $p=0,2$ inner soln could be a boundary layer.

p odd



interior layer

p even



boundary layer

General Remarks

$$\varepsilon y'' + a(x)y' + b(x)y = 0 \qquad a(x) = a'(\bar{x})(x-\bar{x}) + \cdots$$

Layer thickness and type depends on T-series of $a(x)$ where $a(\bar{x}) = 0$.

$a(x) = \sin x \sim x$ — Interior Layer @ $x=0$

$a(x) = \cos x \sim \frac{(x-\pi/2)^2}{2}$ — B-Layer. @ $x = \pi/2$.

EXAMPLE **Different B-Layer Thickness**

(1) $$\varepsilon y'' + x y' - y = x \qquad y(0) = y(1) = 1$$

Find an asymptotic approximation having a B-Layer at $x=0$

OUTER SOLUTION

$$x y_0' - y_0 = x$$

has the general solution

$$y_0(x) = x \ln x + c_1 x$$

Note that $\nexists\, c_1$ s.t. $y_0(0)=1$ satisfies left B.C. Hence a layer must exist at $x=0$ and

(2) $$y_0(x) = x\ln x + x \qquad y_0(1) = 1$$

INNER PROBLEM

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x}{\varepsilon^\beta} \qquad \beta > 0$$

yields

$$\underset{①}{\varepsilon^{1-2\beta} Y''} + \underset{②}{\underbrace{X Y' - Y}} = \underset{③}{\varepsilon^\beta X}$$

Seek a dominant balance s.t. $Y_0(X)$ can be matched to $y_0(x)$.

Note ② $\gg$ ③ for any choice $\beta > 0$.

Were ② $\gg$ ①, $X Y_0' - Y_0 = 0 \Rightarrow Y_0(X) = AX$ can't satisfy B.C. at $x=0$.

Conclude ① ~ ② and

$$(3) \qquad Y'' + X Y' - Y = \varepsilon^{1/2} X \qquad \beta = \tfrac{1}{2}$$

Thus $Y_0(X)$ must solve

$$(4) \qquad Y_0'' + X Y_0' - Y_0 = 0 \qquad Y_0(0) = 1$$

Without symbollic manipulators the solution of (4) can be found by noting

$$Y_0^{(1)}(X) = X$$

is a solution and then use variation of parameters to find $\phi(X)$ s.t.

$$Y_0^{(2)}(X) = X\,\phi(X)$$

is also a solution. After considerable calculations

$$Y_0(X) = e^{-X^2/2} + X\left(B + \sqrt{\tfrac{\pi}{2}}\,\mathrm{erf}\left(\tfrac{X}{\sqrt{2}}\right)\right) \qquad ④$$

where constant B found by matching. Note, in particular, ④ must be bounded as $X \to \infty$.

Matching
Is there a $B \in \mathbb{R}$ s.t.

$$M = \lim_{x \to 0} y_0(x) = \lim_{X \to \infty} Y_0(X)$$

Since $\operatorname{erf}(z) = 1 + o(1)$ as $z \to \infty$, and considering the term ④

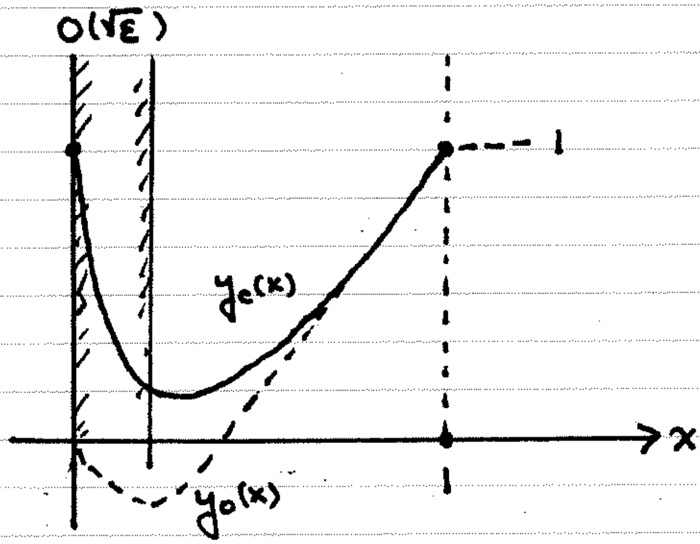
$$B = -\sqrt{\frac{\pi}{2}}$$

so that $M = 0$ and

$$Y_0(X) = e^{-X^2/2} + \sqrt{\frac{\pi}{2}}\, X \left(\operatorname{erf}\left(\frac{X}{\sqrt{2}}\right) - 1 \right)$$

Composite solution

$$y_c(x) = y_0(x) + Y_0\left(\frac{x}{\sqrt{\varepsilon}}\right)$$



Variation of parameters - addendum

Let $u_1(x)$ be any solution of

$$L(u) = u'' + a(x)u' + b(x)u = 0$$

Seek a second independent solution

$$u_2(x) = \phi(x)\,u_1(x)$$

Substituting this into $L(u_2) = 0$ gives

$$\underbrace{u_1\phi'' + (2u_1' + au_1)\phi'}_{\text{need this to vanish}} + \phi \, \cancelto{0}{L(u_1)} = 0$$

Thus $v(x) = \phi'(x)$ is a solution of the first order eqn

$$v' + p(x)\,v = 0 \qquad p(x) = a(x) + \frac{2u_1'}{u_1}$$

which is easily solved so

$$u_2(x) = u_1(x)\int^x v(s)\,ds$$

Corner Layer Example

$$(1) \quad \varepsilon y'' + x y' - y = 0 \qquad x \in (-1, 1)$$

$$(2) \quad y(-1) = 1 \qquad y(1) = 2$$

Here $a(x) = x$ and (1)-(2) has a turning point $\bar{x} = 0$

$$a(\bar{x}) = 0 \qquad \bar{x} = 0$$

No boundary layers

$$y(x, \varepsilon) = Y(X, \varepsilon) \qquad X \equiv \frac{x - \bar{x}}{\varepsilon}$$

Regardless of $\bar{x} = -1, 1$

$$Y_0(X) = A + Be^{-\bar{x} X}$$

For both $\bar{x} = -1$ and $\bar{x} = +1$ the leading inner solution becomes unbounded in the outer region. For instance, $\bar{x} = -1$

$$\lim_{X \to \infty} Y_0(X) = \lim_{X \to \infty} (A + Be^{X}) = \infty$$

Conclude, no boundary layers.

Outer solution(s)

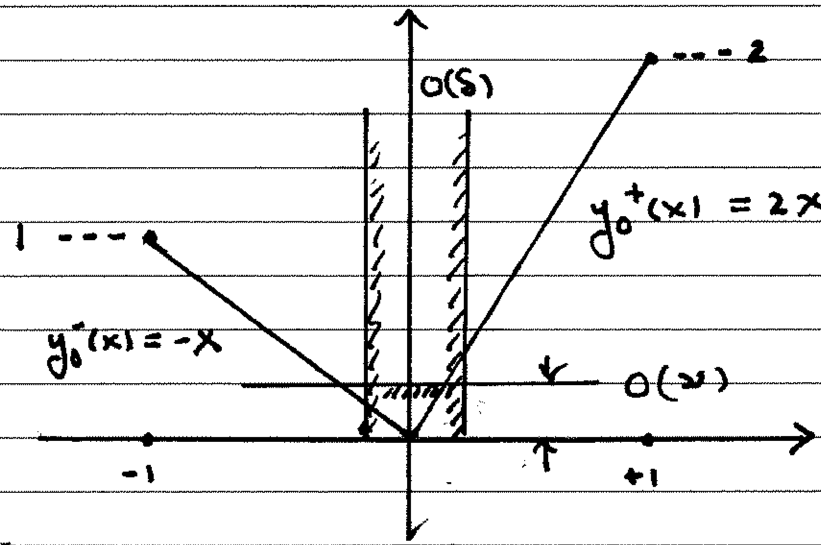
$$x y_0' - y_0 = 0$$

has the general solution $y_0(x) = Ax$.

Independent of the choice of A

$$\lim_{x \to 0^-} y_0(x) = 0 = \lim_{x \to 0^+} y_0(x)$$

so we expect a corner layer



$$y_0(x) = \begin{cases} -x & x < 0 \\ 2x & x > 0 \end{cases}$$

A corner layer at $x = 0$ exists with

$$x = O(\delta) \qquad y = O(\nu)$$

Note: $y_0 \in C(\Omega)$ but $y_0 \notin C^1(\Omega)$ where $\Omega = [-1, 1]$

Inner expansion near corner layer.

Sharp layer near $(\bar{x}, \bar{y})$ where $\bar{y} = y_0(\bar{x})$

$$\mathbb{X} = \frac{x - \bar{x}}{\delta(\varepsilon)} \qquad \delta(\varepsilon) \ll 1$$

In the layer the change in y is $O(\nu)$ over $x = O(\delta)$. We don't know either $\delta \ll 1$ or $\nu \ll 1$.

(1) $y(x, \varepsilon) = \mathbb{Y}(\mathbb{X}, \varepsilon) = y_0(\bar{x}) + \nu(\varepsilon)\,\mathbb{Y}_1(\mathbb{X}) + o(\nu)$

Here $\mathbb{Y}_0(\mathbb{X}) = y_0(\bar{x})$ is constant in (1)

The (exact) equation for $\mathbb{Y}$ is

(2) $$\frac{\varepsilon}{\delta^2}\mathbb{Y}'' + \mathbb{X}\mathbb{Y}' - \mathbb{Y} = 0$$

Must choose

$$\delta(\varepsilon) = \varepsilon^{1/2}$$

Then we get an ODE for $\mathbb{Y}_1(\mathbb{X})$ which essentially is equivalent to solving the original problem:

(3) $$\mathbb{Y}_1'' + \mathbb{X}\mathbb{Y}_1' - \mathbb{Y}_1 = 0$$

The general solution of Inner Problem:

$$Y_1(X) = \underbrace{c_1 X}_{\text{unbnd in } X} + c_2 \underbrace{\left(2e^{-\frac{1}{2}X^2} + \sqrt{2\pi}\,\mathrm{erf}\left(\tfrac{X}{\sqrt{2}}\right)\right)}_{\text{bounded in } X}$$

<u>Matching to $O(1)$</u>

Prandtl matching condition satisfied

$$\lim_{x\to 0} y_0(x) = 0 = y_0(\bar{x}) = \lim_{X\to\pm\infty} Y_0(X)$$

recalling $Y_0(X)$ is constant, i.e. $Y_0(X) \equiv y_0(\bar{x})$

More formally, for $\eta \ll 1$.

$$\lim_{\substack{\varepsilon\to 0 \\ z = x/\eta(\varepsilon)\text{ fixed}}} \left(y_0(\eta z) - Y_0\left(\tfrac{\eta z}{\varepsilon}\right)\right)$$

$$= \lim_{\substack{\varepsilon\to 0 \\ z\text{ fixed}}} \left(y_0(\eta(\varepsilon) z) - y_0(\bar{x})\right)$$

$$= 0$$

Matching to $O(\nu)$ (ν still not known)

outer $\quad y(x,\varepsilon) \sim y_0(x) + \varepsilon\, y_1(x) + O(\varepsilon^2)$

inner $\quad Y(X,\varepsilon) \sim y_0(\bar{x}) + \nu(\varepsilon)\, Y_1(X) + o(\nu)$

Guess one term outer soln matches two term inner to $O(\nu)$

Intermediate variable $\delta(\varepsilon) = \varepsilon^{1/2}$

$$z = \frac{x}{\eta(\varepsilon)} \qquad \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1$$

Matching conditions are

$$(4) \qquad \boxed{\lim_{\substack{\varepsilon \to 0 \\ z\text{-fixed}}} \frac{\left(y_0^{\pm}(\eta z) - \nu(\varepsilon)\, Y_1\left(\frac{\eta z}{\delta}\right)\right)}{\nu(\varepsilon)} = 0}$$

Need a large X approximation of $Y_1(X)$.

From Tables we have

$$(5) \qquad \operatorname{erf}(X) = \pm 1 - \frac{1}{\sqrt{\pi}\, X} e^{-X^2} + \frac{1}{2\sqrt{\pi}\, X^3} e^{-X^2} + \cdots$$

as $X \to \pm\infty$

Define

$$\frac{M_{01}^{+}}{\nu(\varepsilon)} = \frac{y_0(x) - \nu(\varepsilon) Y_1(X)}{\nu(\varepsilon)} \qquad \underline{\underline{x > 0}}$$

Given previous defns and (5)

$$\frac{M_{01}^{+}}{\nu(\varepsilon)} = \underbrace{\frac{\varepsilon^{1/2}}{\nu} \overset{\Downarrow}{2X} - (\sqrt{2\pi}\, C_2 + C_1) \overset{\Downarrow}{X}}_{\text{need to vanish}} + O(\underbrace{X^{-1} e^{-X^2}}_{\text{T.S.T.}})$$

<u>Conclude</u>

$$\begin{Vmatrix} (a) & \nu(\varepsilon) = \varepsilon^{1/2} \\ (b) & 2 = \sqrt{2\pi}\, C_2 + C_1 \end{Vmatrix}$$

A final detail

$$X^2 = \frac{z^2 \eta^2}{\varepsilon} = |\ln \varepsilon| \iff z\eta = |\varepsilon \ln \varepsilon|^{1/2}$$

Thus the term above is T. S. T. if

$$|\varepsilon \ln \varepsilon|^{1/2} \ll \eta \ll 1$$

Also the terms that "matched" above are indicated by $\Downarrow$. Here, for $x > 0$,

$$M_+ = 2X$$

needed for composite. soln.

Similar expansions for left overlap domain yields

$$(c) \qquad -1 = \sqrt{2\pi}\, C_2 - C_1$$

with the match term

$$M_- = -\bar{X}$$

Solving (b)-(c) yields

$$C_1 = 3/2 \qquad C_2 = \frac{\sqrt{2}}{4\sqrt{\pi}}$$

completing inner problem.

<u>Composite Solution</u> $\quad \nu(\varepsilon) = \varepsilon^{1/2}$

$$y_c(x,\varepsilon) = \begin{cases} y_0^+(x) + \nu(\varepsilon)\,(\bar{Y}_1(\bar{X}) - M_+) & x > 0 \\ y_0^-(x) + \nu(\varepsilon)\,(\bar{Y}_1(\bar{X}) - M_-) & x < 0 \end{cases}$$

Moral : Corner layers are bad news from an analysis perspective. Often harder than original problem.

However, outer solns are easy!

EXAMPLE Interior singularity

$$(1)\quad \boxed{\begin{array}{ll} \varepsilon y'' + 2xy' + (2+\varepsilon x^2)y = 0 & x \in (-1,1) \\ y(-1) = 2 \qquad y(1) = -2 & \end{array}}$$

Solution $y(x)$ is odd

Easy to see the problem (1) including B.C. is invariant under $(x,y) \to (-x,-y)$.

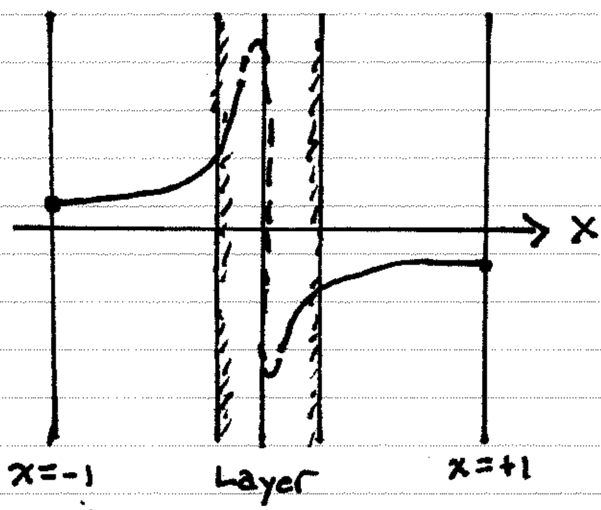
Alternately if $y(x)$ is the solution of (1) for $x>0$, define $v(z) = -y(-z)$ for $z<0$. With $z=-x$, can verify $v(z)$ solves (1) for $z<0$.

Outer Problem $\quad a(x) = 2x \qquad \bar{x} = 0$ turning point

$$(2)\qquad 2xy_0' + 2y_0 = 0$$

has a general solution $y_0(x) = \frac{A}{x}$, $x \neq 0$.

The outer solution(s) is <u>unbounded</u> in layer



$$y_0^+(x) = -\frac{2}{x} \qquad x>0$$

$$y_0^-(x) = \frac{2}{x} \qquad x<0$$

---- inner soln

Prandtl Matching Fails

Since $y_0(x)$ is unbounded $Y(X, \varepsilon)$ should be too. Also, Prandtl matching fails since, in particular,

$$\lim_{x \to 0} y_0(x) \qquad \text{D.N.E.}$$

Inner Problem

$$(1) \qquad y(x, \varepsilon) = \varepsilon^{-\alpha} Y(X, \varepsilon) \qquad X = \frac{x}{\varepsilon^{\beta}}$$

where $\alpha, \beta > 0$ is an assumed asymptotic behavior. That such an inner soln matches for some α, β validates the assumption. For any α

$$(2) \qquad \varepsilon^{1-2\beta} Y'' + 2XY' + (2 + O(\varepsilon))Y = 0$$

$$① \qquad ② \qquad ③$$

If ② ~ ③ we repeat the outer problem. Choose $\beta = \frac{1}{2}$ so ① ~ ② and

$$(3) \qquad Y_0'' + 2XY_0' + 2Y_0 = 0$$

and

$$y(x, \varepsilon) = \varepsilon^{-\alpha} Y_0(X) + o(\varepsilon^{-\alpha})$$

Notice the remainder though asymptotically smaller than $\varepsilon^{-\alpha}$, may still be large.

Solution of (3) is

$$Y_0(X) = a e^{-X^2} + b e^{-X^2}\int_0^X e^{t^2}\,dt$$

Since $y(x,\varepsilon)$ is odd, $Y_0(X)$ must be too, so $a=0$

$$(4) \qquad Y_0(X) = b e^{-X^2}\int_0^X e^{t^2}\,dt \qquad b \in \mathbb{R}$$

<u>Matching</u>

wlog need only match $Y_0(X)$ to $y_0^+(x) = -\frac{2}{x}$, $x>0$
Thus, need large X expansion for $Y_0(X)$.

Use $u = \frac{1}{2t}$ and $v = e^{t^2}$ and integrate (4) by parts.

$$Y_0(X) = \frac{b}{2X} + \underbrace{\tfrac{1}{2} b \exp(-X^2)\int_\delta^X \frac{e^{t^2}}{t^2}\,dt}_{f(X)} + \text{T.S.T}$$

Claim $f = O\left(\frac{1}{X^3}\right)$ as $X \to \infty$. Given claim

$$Y_0(X) = \frac{b}{2X} + O\left(\frac{1}{X^3}\right) \quad \text{as } X \to \infty$$

Claim is verified by showing

$$\lim_{X\to\infty} X^3 f(X) = \tfrac{1}{2}$$

Formal Matching

For $x = \eta z$, $\eta \ll 1$ intermediate variable need to show

$$(1) \quad \lim_{\substack{\varepsilon \to 0^+ \\ z > 0}} \frac{\left(y_0(\eta z) - \varepsilon^{-\alpha} Y_0\left(\frac{\eta z}{\varepsilon^{1/2}}\right)\right)}{\nu(\varepsilon)} = 0$$

for some $\nu(\varepsilon) \ll \varepsilon^{-\alpha}$ (in case difference large)

Let M_{00} be numerator of (1). Given $y_0^+(x) = -\frac{2}{x}$

$$M_{00} = \overset{\text{out}}{-\frac{2}{\eta z}} - \varepsilon^{-\alpha}\left(\overset{\text{inn.}}{\frac{b}{2X}} + f(X)\right)$$

$$M_{00} = \underbrace{-\frac{2}{\eta z} - \frac{b\,\varepsilon^{\frac{1}{2}-\alpha}}{2\eta z}}_{\substack{\text{leading matching} \\ \text{terms cancel} \\ \text{for all } \nu \text{ if}}} \underbrace{- \varepsilon^{-\alpha} f\left(\frac{\eta z}{\varepsilon^{1/2}}\right)}_{\substack{\text{for appropriate} \\ \text{restrictions on} \\ \eta(\varepsilon) \text{ is small} \\ \text{(Difficult)}}}$$

$$\alpha = \frac{1}{2} \qquad b = -4$$

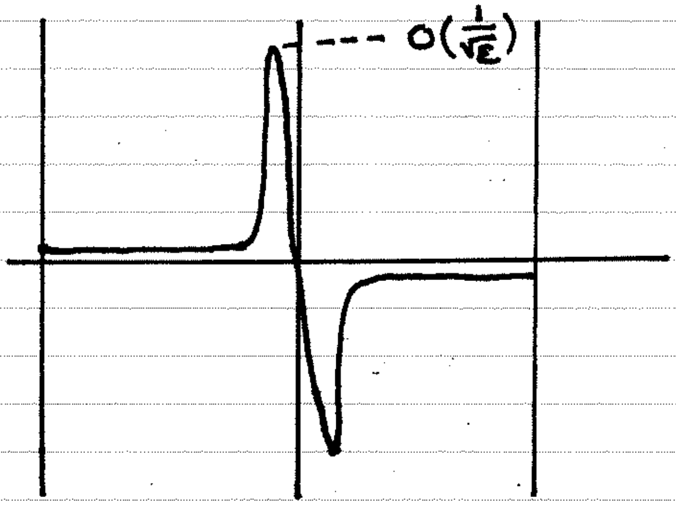
$$\boxed{\varepsilon^{-\alpha} Y_0(X) = -4\varepsilon^{-1/2} \exp(-X^2) \int_0^X e^{t^2}\,dt}$$

Composite Solution (x > 0 wlog)

$$y_c(x,\varepsilon) = y_0(x) + \varepsilon^{-\alpha} Y_0(X) - \text{matching terms in overlap region}$$

The matching term is $y_0(x)$ here, so

$$y_c(x,\varepsilon) = -4\varepsilon^{-1/2} \exp\left(\frac{-x^2}{\varepsilon}\right) \int_0^{x/\sqrt{\varepsilon}} e^{t^2}\,dt .$$



dipole pulse.

Looks like $\delta'(x)$, delta fn.

Nonlinear Interior Layer example

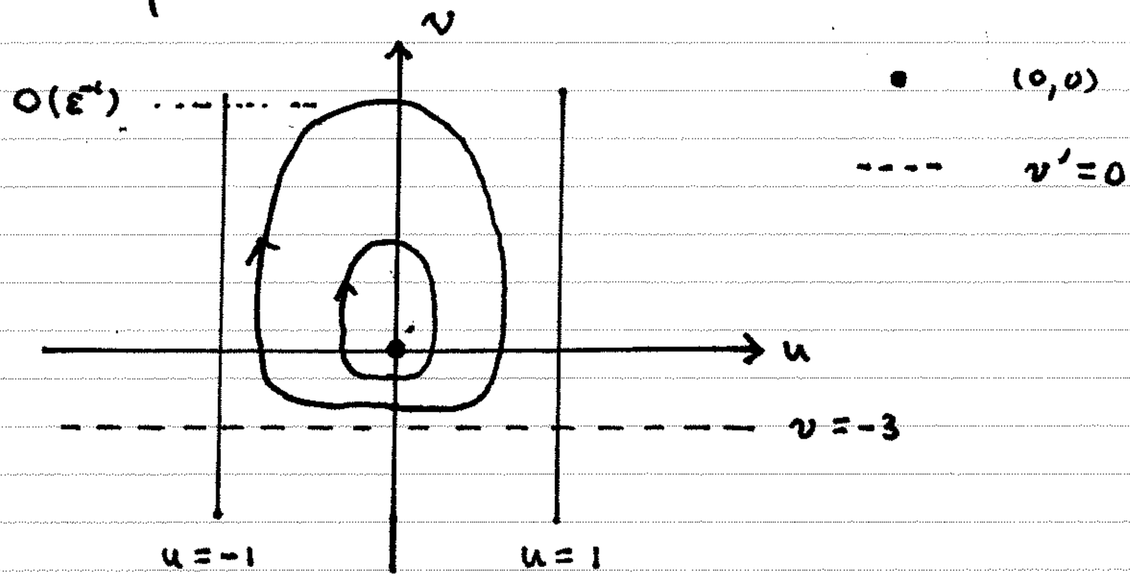
$$\boxed{\begin{gathered} \varepsilon y'' + y(y'+3) = 0 \\ y(0) = -1 \qquad y(1) = 1 \end{gathered}}$$

As a first order system

$$u' = v$$

$$v' = -\frac{1}{\varepsilon} u(v+3)$$

Has a sole equilibria $(0,0)$ which is a center.
Phase portrait



Boundary condition curves $u = \pm 1$ drawn.

Solution starts on $u=-1$ and ends on $u=+1$ over $x \in (0,1)$

Outer Solution

$$y_0(y_0' + 3) = 0$$

$y_0(x) = 0$ not possible since $(0,0)$ equilibria:

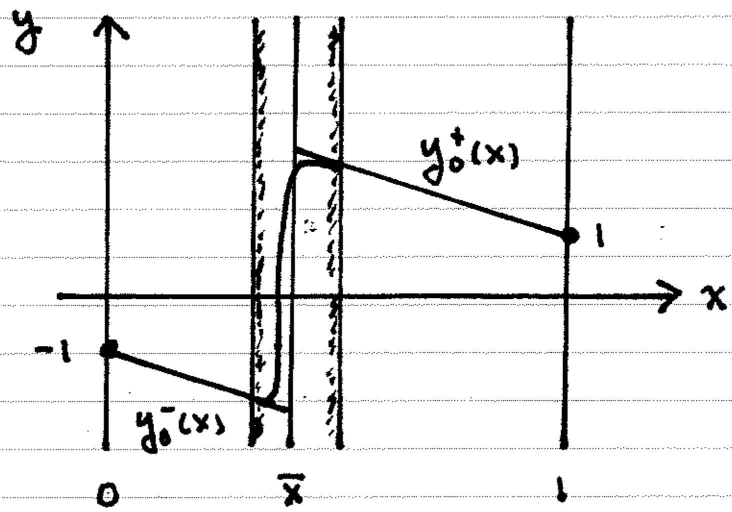
$$y_0^{\pm}(x) = -3x + c_{\pm}$$

Using boundary conditions

$$y_0^-(x) = -3x - 1 \qquad y_0^-(0) = -1 \qquad x < \bar{x}$$

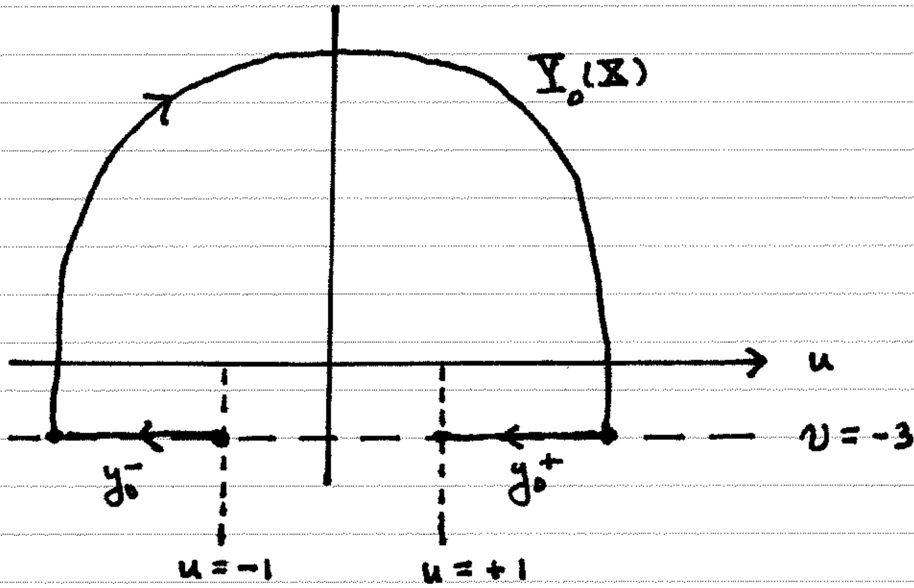
$$y_0^+(x) = -3x + 4 \qquad y_0^+(1) = 1 \qquad x > \bar{x}$$

Assuming an interior layer at $x = \bar{x}$ where $\bar{x}$ is to be determined.



Inner solution shown with layer thickness $\delta(\varepsilon)$

In phase space the solution would look like



Inner Problem

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x - \bar{x}}{\delta} \qquad \delta \ll 1$$

Leads to

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y Y' + 3Y = 0$$

terms must balance

Thus the B-Layer thickness is $\delta(\varepsilon) = \varepsilon$ and

$$\boxed{Y_0'' + Y_0 Y_0' = 0}$$

To solve inner problem note it is equivalent to

$$\frac{d}{d\bar{X}}\left(Y_0' + \frac{1}{2} Y_0^2 \right) = 0$$

Term in parenthese is constant which leads to a separable first order ODE:

$$\boxed{Y_0(\bar{X}) = c_1 \tanh\left(\frac{c_1}{2}(\bar{X} - c_2)\right)}$$

Matching

$$M_0^- = \lim_{x \to \bar{x}} y_0^-(x) = \lim_{\bar{X} \to -\infty} Y_0(\bar{X})$$

$$M_0^+ = \lim_{x \to \bar{x}} y_0^+(x) = \lim_{\bar{X} \to \infty} Y_0(\bar{X})$$

yields two equations for $\bar{x}$ and c_1

$$M_0^- = -3\bar{x} - 1 = -c_1$$

$$M_0^+ = -3\bar{x} + 4 = c_1$$

Solving

$$\boxed{\bar{x} = \frac{1}{2} \qquad c_1 = \frac{5}{2}}$$

Have found layer position but not c_2. Also

$$M_0^- = -\frac{5}{2} \qquad M_0^+ = \frac{5}{2}$$

Value of c_2

$$\bar{Y}_0(\bar{X}) := \frac{5}{2}\tanh\left(\frac{5}{4}(\bar{X}-c_2)\right)$$

$$\bar{Y}_0(\bar{X}) = \frac{5}{2}\tanh\left(\frac{5}{4}\left(\frac{x-\frac{1}{2}}{\varepsilon}-c_2\right)\right)$$

$$\bar{Y}_0(\bar{X}) = \frac{5}{2}\tanh\left(\frac{5}{4}\frac{(x-\bar{x}(\varepsilon))}{\varepsilon}\right)$$

where

$$\bar{x}(\varepsilon) = \frac{1}{2} + c_2\varepsilon$$

so c_2 is $O(\varepsilon)$ correction to layer location.
To leading order we may assume $c_2 = 0$ w.l.o.g.

$$\bar{Y}_0(\bar{X}) = \frac{5}{2}\tanh\left(\frac{5}{4}\bar{X}\right)$$

Composite solution

$$y_c^{\pm}(x,\varepsilon) = y_0^{\pm}(x) + \bar{Y}_0(\bar{X}) - M^{\pm} \qquad \text{(match terms)}$$

is piecewise defined

$$y_c(x,\varepsilon) = \begin{cases} -3x-1 + \frac{5}{2}\tanh\left(\frac{5(x-\frac{1}{2})}{4\varepsilon}\right) + \frac{5}{2} & x < \frac{1}{2} \\ -3x+4 + \frac{5}{2}\tanh\left(\frac{5(x-\frac{1}{2})}{4\varepsilon}\right) - \frac{5}{2} & x \geq \frac{1}{2} \end{cases}$$