Asymptotic approximations - introduction

Use the smallness (or largeness) of a parameter (or independent variable) to find an approximation u_ε that is either near in an absolute or relative sense.

EXAMPLE (Algebraic) $0 < \varepsilon \ll 1$

$$x^3 + \varepsilon x - 1 = 0 \qquad x = 1 - \tfrac{1}{3}\varepsilon + \cdots$$

$$x^2 + \varepsilon x - \varepsilon = 0 \qquad x = \sqrt{\varepsilon} - \tfrac{1}{2}\varepsilon + \cdots$$

$$\varepsilon x^2 + x - 1 = 0 \qquad x = -\frac{1}{\varepsilon} - 1 + \cdots$$

EXAMPLE (Integral)

$$Ei(x) \equiv \int_x^\infty \frac{e^{-z}}{z}\,dz$$

occurs in optics. Large x approximations sometimes need. If one defines

$$\phi(x) = \frac{e^x}{x}$$

then one can show

$$\lim_{x\to\infty} \frac{Ei(x)}{\phi(x)} = 1$$

so that $\phi(x)$ is an approximation in a relative (or asymptotic) sense

EXAMPLE <u>Slowly varying oscillator</u>

$$y'' + \omega^2(\varepsilon t)\, y = 0 \qquad \omega(z) = 1 + z$$

Slow varying $\omega(\varepsilon t)$ affects amplitude and phase over long times

EXAMPLE <u>Perturbed eigenvalue problems</u>

$$A(\varepsilon)\, x = \lambda x \qquad A \in \mathbb{R}^{n \times n}$$

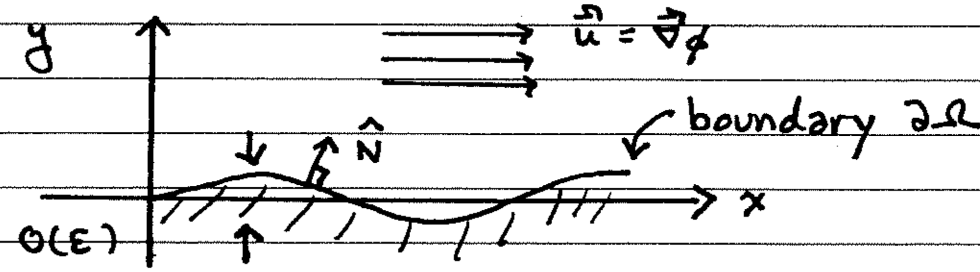
$$L_\varepsilon u = \lambda u \qquad u \in D$$

Two specific examples of the latter

$$u_{xx} + (1 + \varepsilon x)\, u = \lambda u \qquad u(0) = u(1) = 0$$

$$\nabla u + \varepsilon u = \lambda u \qquad u\big|_{\partial\Omega} = 0$$

EXAMPLE <u>Perturbed Boundaries</u>



Irrotational, incompressible, inviscid fluid flow

$$\nabla^2 \phi = 0 \qquad x \in \Omega$$

$$\frac{\partial \phi}{\partial n} = 0 \qquad x \in \partial\Omega$$

EXAMPLE Perturbed Hamiltonians

For some systems the time dependence of the Hamiltonian is weak: slight planetary assymetries, stars lose mass, ...

$$H = H(q, p, \tau) \qquad \tau = \varepsilon t$$

Seek long time approximations to Hamilton Eqns

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

EXAMPLE Difference eqns

$$a_{n+1} = (n+1)\, a_n \qquad a_1 = 1$$

has solution $a_n = n!$. Stirling's formula for large n approximations is

$$a_n \sim \sqrt{2\pi}\; n^{n-\frac{1}{2}} e^{-n} \left(1 + \frac{1}{12n}\right) \equiv A_n$$

For instance

n	a_n	A_n
6	720	719.955
20	*	8×10^{-6} relative error

EXAMPLE **Small displacement approximations**

Recall the minimal surface area problem

$$J(u) = \int_{\Omega} \sqrt{1 + u_x^2 + u_y^2}\, dA$$

Suppose

$$u(x, \varepsilon) = \varepsilon g(x) \qquad x \in \Omega$$

where g is smooth. The EL-eqns are

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right) = 0$$

Letting

$$u = \varepsilon v = \varepsilon v_1 + \varepsilon^2 v_2 + \cdots$$

yields a leading problem

$$\nabla^2 v_1 = 0 \qquad v_1 = g \quad \text{on } \partial\Omega$$