Kuzmak-Luke Method

Multiple scales method applicable to IVP:

(1) $y'' + g(y,\tilde{t}) + \varepsilon h(y,\dot{y},\tilde{t}) = 0 \quad , \quad \tilde{t} = \varepsilon t$

(2) $y'' + g(y,0) = 0$ has periodic solutions

(3) h odd in $\dot{y}$

Asymptotic form assumption

(4) $y(t,\varepsilon) = Y(\tau,\tilde{t};\varepsilon) = Y_0(\tau,\tilde{t}) + \varepsilon Y_1(\tau,\tilde{t}) + O(\varepsilon^2)$

(5) $\dfrac{d\tau}{dt} = \omega_0(\tilde{t}) + \varepsilon\omega_1(\tilde{t}) + O(\varepsilon^2)$

Require Y_i to be 2π-periodic in τ and expansion to be consistent.

Straining functions ω_i and other functions of $\tilde{t}$ to be determined by method.

(6) $\tau = \dfrac{\theta(\tilde{t})}{\varepsilon} + \phi_0(\tilde{t}) + O(\varepsilon)$

$$\theta(\tilde{t}) = \int_0^{\tilde{t}} \omega_0(s)\,ds$$

$$\phi_0(\tilde{t}) = \phi_0(0) + \int_0^{\tilde{t}} \omega_1(s)\,ds$$

Governing equations

$$(7) \quad N[Y_0] \equiv \omega_0^2 \frac{\partial^2 Y_0}{\partial \tau^2} + g(Y_0, \tilde{t}) = 0 \qquad O(1)$$

$$(8) \quad L[Y_1] \equiv \omega_0^2 \frac{\partial^2 Y_1}{\partial \tau^2} + g_y(Y_0, \tilde{t}) Y_1 = r_1 \qquad O(\varepsilon)$$

where

$$(9) \quad r_1 = -2\omega_0 \omega_1 \frac{\partial^2 Y_0}{\partial \tau^2} - 2\omega_0 \frac{\partial^2 Y_0}{\partial \tau \, \partial \tilde{t}} - \omega_0' \frac{\partial Y_0}{\partial \tau} - h(Y_0, \omega_0 \frac{\partial Y_0}{\partial \tau}, \tilde{t})$$

Solution to O(1)

$$(10) \quad \frac{1}{2}\omega_0^2 \left(\frac{\partial Y_0}{\partial \tau}\right)^2 + V(Y_0, \tilde{t}) = E_0(\tilde{t}) \qquad \text{"energy"}$$

where

$$(11) \quad V(Y_0, \tilde{t}) \equiv \int_0^{Y_0} g(\eta, \tilde{t})\, d\eta \qquad \text{"potential energy"}$$

Integrating (10) and inverting

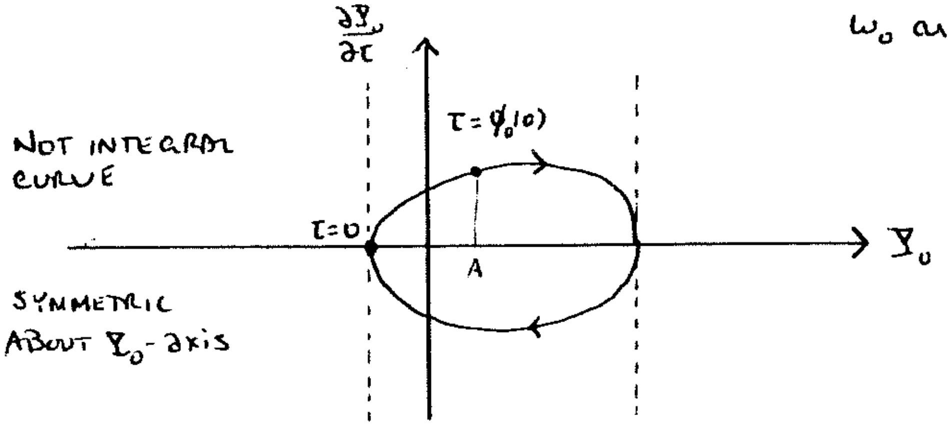
$$Y_0(\tau, \tilde{t}) = f(\tau + \lambda_0(\tilde{t}), E_0(\tilde{t}), \omega_0(\tilde{t}), \tilde{t})$$

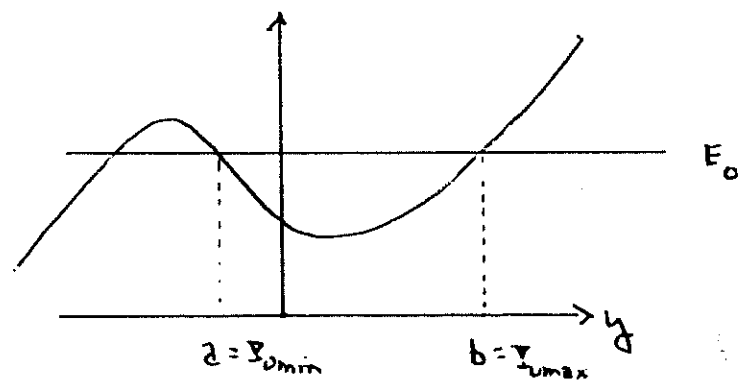
Since a phase variation ϕ_0 included in τ we can wlog set $\lambda_0 = 0$.

Next, seek period of Y_0 in τ for fixed $\tilde{t}$

Period of Y_0 for $\tilde{t}$ fixed

Since $\tilde{t}$ fixed, E_0 and ω_0 are fixed also.





$\tilde{t}$ fixed

INITIAL CONDITIONS $\quad t = 0 \iff \tau = \phi(0), \quad \tilde{t} = 0$

$$y(0) = A \quad \Rightarrow \quad Y_0(\phi_0(0), 0) = A$$

$$\dot{y}(0) = B \quad \Rightarrow \quad \frac{\partial Y_0}{\partial \tau}(\phi_0(0), 0) = \frac{B}{\omega_0(0)}$$

* CHOOSE $\phi_0(0)$ SO THAT $Y_0(0, -\varepsilon\phi_0(0)) = Y_{0min} = a$

* SYMMETRY ABOUT $Y_0 \Rightarrow Y_0(\tau, \tilde{t})$ EVEN IN τ

DETERMINE $\Upsilon_{0\,min}$, $\Upsilon_{0\,max}$ WHERE $\frac{\partial \Upsilon_0}{\partial \tau} = 0$ FOR FIXED $\hat{t}$

$$V(a, \tilde{t}) = E_0 = V(b, \tilde{t})$$

inversion implies

$$\Upsilon_{0\,min} = \Upsilon_{0\,min}(E_0, \tilde{t})$$
$$\Upsilon_{0\,max} = \Upsilon_{0\,max}(E_0, \tilde{t})$$

INTEGRATE ENERGY EQUATION WITH $\phi_0(0)$ DEFN YIELDS

$$\tau = \operatorname{sgn}\left(\frac{\partial \Upsilon_0}{\partial \tau}\right) \omega_0 \int_{\Upsilon_{0\,min}(E_0, \tilde{t})}^{\Upsilon_0(\tau, \hat{t})} \frac{d\eta}{\sqrt{2(E_0 - V(\eta, \tilde{t}))}}$$

From which we obtain the period of Υ_0 in τ

$$P = P(E_0, \omega_0; \tilde{t}) = 2\omega_0 \int_{\Upsilon_{0\,min}(E_0, \tilde{t})}^{\Upsilon_{0\,max}(E_0, \tilde{t})} \frac{d\eta}{\sqrt{2(E_0 - V(\eta, \tilde{t}))}}$$

STRAINING OF τ

Require a fixed period, say 2π, in τ independent of $\tilde{t}$.

$$P(E_0, \omega_0, \hat{t}) = 2\pi$$

yields

$$\omega_0 = \Omega(E_0, \tilde{t}) = \pi \left\{ \int_{\Upsilon_{0\,min}}^{\Upsilon_{0\,max}} \frac{d\eta}{\sqrt{2(E_0 - V(\eta, \tilde{t}))}} \right\}^{-1}$$

SUMMARY TO O(1)

$$Y_0(\tau, \tilde{t}) = p(\tau, E_0(\tilde{t}), \tilde{t}) \equiv f(\tau, E_0(\tilde{t}), \Omega(E_0(\tilde{t}), \tilde{t}), \tilde{t})$$

$V(a, \tilde{t}) = E_0 = V(b, \tilde{t})$ determines $Y_{0\,min,max}$

$\omega_c(\tilde{t}) = \Omega(E_0(\tilde{t}), \tilde{t})$ known

$\phi_0(0)$ chosen so Y_0 is 2π-periodic even function in τ.

Digress: why fixed period in τ

Needed to retain uniformity for large t. Suppose that P were not constant, then

$$Y_0(\tau, \tilde{t}) = Y_0(\tau + nP(\tilde{t}), \tilde{t}) \equiv \bar{Y}_0(\zeta, \tilde{t})$$

A computation in Kevorkian/Cole

$$\frac{\partial \bar{Y}_0}{\partial \tilde{t}} = \frac{\partial Y_0}{\partial \tilde{t}}(\tau + nP, \tilde{t}) = \frac{\partial Y_0}{\partial \tilde{t}}(\tau, \tilde{t}) - n \frac{dP}{d\tilde{t}} \frac{\partial Y_0}{\partial \tau}(\tau, \tilde{t})$$

which implies $\frac{\partial Y_0}{\partial \tilde{t}}$ gets large with n unless $\frac{dP}{d\tilde{t}} = 0$.

General Solution of $O(\varepsilon)$ equation

$$L[Y_1] = \omega_0^2 \frac{\partial^2 Y_1}{\partial \tau^2} + g_y(Y_0, \tilde{t})\, Y_1 = r_1$$

First note

$$Y_0 = p(\tau, E_0(\tilde{t}), \tilde{t})$$ is 2π-periodic even soln of $\mathcal{N}[\cdot] = 0$

Hence

$$\frac{\partial}{\partial \tau} \mathcal{N}[p] = \frac{\partial}{\partial \tau} \left\{ \omega_0^2 \frac{\partial^2 p}{\partial \tau^2} + g(p, \tilde{t}) \right\} = 0$$

Expanding yields

$$L\left[\frac{\partial p}{\partial \tau}\right] = 0$$

Thus $\frac{\partial p}{\partial \tau}$ is a homogeneous solution of $O(\varepsilon)$ problem.

Luke originally used $\frac{\partial f}{\partial E_0}$ as a second solution but Bourland and Haberman demonstrated another general form

$$q(\tau, E_0, \tilde{t}) = \Omega(E_0, \tilde{t}) \frac{\partial p}{\partial E_0} + \frac{\partial \Omega}{\partial E_0} \cdot \tau \cdot \frac{\partial p}{\partial \tau}$$

$$L[q] = 0$$

q is a second <u>even</u> in τ solution but not periodic

Also, showed Wronskian

$$W = \frac{\partial p}{\partial \tau}\frac{\partial q}{\partial \tau} - q\frac{\partial^2 p}{\partial \tau} = \cdots = \frac{1}{\Omega} \neq 0.$$

Thus $\frac{\partial p}{\partial \tau}$ is only periodic solution of $L[\Psi_1] = 0$.

Using an alternative theorem, since L is self adjoint, $\frac{\partial p}{\partial \tau}$ must be orthogonal to rhs of $L[\Psi_1] = 0$.

(1) $$L[\Psi_1] = r_1 = r_{1\,even} + r_{2\,odd}$$

where (after much simplification)

(2) $$r_{1e} = -2\Omega\omega_1 \frac{\partial^2 p}{\partial \tau^2}$$

(3) $$r_{1o} = -2\Omega\left[\frac{\partial^2 p}{\partial \tau \partial E_0}\frac{dE_0}{d\tilde{t}} + \frac{\partial^2 p}{\partial \tau \partial \tilde{t}}\right] - \frac{d\Omega}{d\tilde{t}}\frac{\partial p}{\partial \tau} - h(p, \Omega\frac{\partial p}{\partial \tau}, \tilde{t})$$

Noting p even, $\frac{\partial p}{\partial \tau}$ odd $\Rightarrow$

(4) $$\langle r_{1e}, \frac{\partial p}{\partial \tau}\rangle = \int_0^{2\pi} (\text{odd} \cdot 2\pi \text{ periodic})\, d\tau = 0$$

automatically. Then, the soluability condition is determined from

(5) $$\langle r_{1o}, \frac{\partial p}{\partial \tau}\rangle = \tilde{\mathcal{F}}\left(\frac{dE_0}{d\tilde{t}}, E_0, \tilde{t}\right) = 0$$

Reduces to

(6) $$\frac{dJ}{d\tilde{t}} + D = 0$$

where

(7) $$J(E_0, \tilde{t}) \equiv \Omega \int_0^{2\pi} \left(\frac{\partial p}{\partial \tau}\right)^2 d\tau = 2 \int_{y_{0min}}^{y_{0max}} \sqrt{2(E_0 - V(\eta, \tilde{t}))}\, d\eta$$

"$J = \frac{E}{\omega}$" is the "average action" and the "dissipation" is

(8) $$D(E_0, \tilde{t}) \equiv \int_0^{2\pi} h\left(p, \Omega \frac{\partial p}{\partial \tau}, \tilde{t}\right) \frac{\partial p}{\partial \tau}\, d\tau$$

SOLVING (6) with initial conditions for $E_0(0)$

(9) $$E_0(0) = \frac{1}{2} \Omega(E_0(0), 0)^2 \cdot \frac{B^2}{\Omega(E_0(0), 0)^2} + V(A, 0)$$

(9') $$E_0(0) = \frac{1}{2} B^2 + V(A, 0)$$

yields $E_0(\tilde{t})$

At this point need $\phi_0(\tilde{t})$ or $\omega_1(\tilde{t})$ to complete solution to $O(1)$ for $t = O(1/\varepsilon)$.

DETERMINATION OF ω_1 (OR $\phi_0(\tilde{t})$)

Using $y(t,\varepsilon) = \Psi(\tau, \tilde{t}; \varepsilon)$ and $\frac{d\tau}{dt} = \omega_0(\tilde{t}) + \varepsilon\omega_1(\tilde{t})$ exact, then the equation for Ψ can be obtained

$$(1) \quad \omega \frac{\partial^2 \Psi}{\partial \tau^2} + g + \varepsilon \left[\frac{d\omega}{d\tilde{t}} \frac{\partial \Psi}{\partial \tau} + 2\omega \frac{\partial^2 \Psi}{\partial \tau \partial \tilde{t}} + h\left(\Psi, \omega \frac{\partial \Psi}{\partial \tau} + \varepsilon \frac{\partial \Psi}{\partial \tilde{t}}, \tilde{t}\right)\right] = O(\varepsilon^2)$$

where $\omega \equiv \omega_0 + \varepsilon\omega_1$. Multiply (1) by $\frac{\partial \Psi}{\partial \tau}$ and use 2π-periodicity of Ψ in τ.
The $O(\varepsilon)$ and $O(\varepsilon^2)$ terms are non zero

$O(\varepsilon)$ contribution (4.4.51) yields periodicity constraint for $\omega_0 = \Omega(E_0, \tilde{t})$.

AFTER MUCH WORK:

$$\omega_1(\tilde{t}) \frac{J_{E_0}}{\Omega_{E_0}} = \frac{\omega_1(0)\, J_{E_0}(E_0(0), 0)}{\Omega_{E_0}(E_0(0), 0)} \exp\left\{ -\int_0^{\tilde{t}} \frac{D_{E_0}(E_0(s), s)}{J_{E_0}(E_0(s), s)}\, ds \right\}$$

4.4 Two Scale Expansions for Strictly Nonlinear Oscillators

In this section we generalize the ideas of two scale expansions to a *strictly* nonlinear second order equation with solutions that are slowly modulated oscillations. Thus, as $\epsilon \to 0$ the equation remains nonlinear.

The basic technique is due to Kuzmak [4.6], who studies a special form of

$$\ddot{y} + g(y, \tilde{t}) + \epsilon h(y, \dot{y}, \tilde{t}) = 0, \tag{4.4.1}$$

where as usual, $\dot{} = d/dt$, $\tilde{t} = \epsilon t$ and $0 < \epsilon \ll 1$. Here h and g are given functions, analytic with respect to each of their arguments. We assume h to be odd in $\dot{y}$ to model dissipation; Kuzmak assumes h to be proportional to $\dot{y}$. The only other restriction is that for $\epsilon = 0$, the reduced nonlinear oscillator

$$\ddot{y} + g(y, 0) = 0 \tag{4.4.2}$$

has periodic solutions. This condition is satisfied whenever the potential $V(y, 0) = \int_0^y g(s, 0)ds$ is concave in some interval $y_1 < y < y_2$, and we restrict attention to oscillations in this interval.

Kuzmak works out the $O(1)$ solution partially; the equation of the slowly varying phase is not derived. In [4.29], Luke studies nonlinear nearly periodic dispersive waves and extends Kuzmak's results to higher order. Mathematically, the solution for such waves essentially reduces to (4.4.1) with $h = 0$, and Luke states that in this case the phase is constant. Bourland and Haberman [4.30] give a careful analysis of (4.4.1) and derive the equation governing the slowly varying phase. In many applications including that of sustained resonance, to be discussed in Sec. 5.3, a more general version of (4.4.1) arises. The damping and restoring force terms also depend on n slowly varying quantities ρ_i, $i = 1, \ldots, n$ with the ρ_i governed by n first order equations of the form $d\rho_i/dt = O(\epsilon)$. This problem is discussed in [4.31].

Here we restrict attention to the simple form (4.4.1) that suffices to illustrate all the essential features. We develop the two scale expansion of the solution based primarily on the approach in [4.30]. A specific example is then worked out in detail.

4.4.1 General Theory

Expansion procedure

We assume that the solution of (4.4.1) can be expressed in the following two scale form

$$y(t; \epsilon) = Y(t^+, \tilde{t}; \epsilon) = Y_0(t^+, \tilde{t}) + \epsilon Y_1(t^+, \tilde{t}) + O(\epsilon^2), \tag{4.4.3}$$

where $E_0(\tilde{t})$ is the slowly varying energy and V is the potential defined by

$$V(Y_0, \tilde{t}) = \int_0^{Y_0} g(\eta, \tilde{t}) d\eta. \qquad (4.4.8b)$$

We next integrate (4.4.8a) and invert the result to express Y_0 in the form

$$Y_0(t^+, \tilde{t}) = f(t^+ + \lambda_0(\tilde{t}), E_0(\tilde{t}), \omega_0(\tilde{t}), \tilde{t}), \qquad (4.4.9)$$

where $\lambda_0(\tilde{t})$ is a slowly varying phase-shift that arises as an additive integration "constant" to t^+. At this point it is useful to note that there is no loss of generality setting $\lambda_0 = 0$ as we have already included an arbitrary phase-shift $\phi_0(\tilde{t})$ in the definition of t^+. In fact, we see that the two unknowns λ_0 and ϕ_0 appear in the solution only in the combination $\lambda_0 + \phi_0$ so one or the other of these two constants may be ignored. In [4.30] λ_0 is set equal to zero and ϕ_0 is retained, whereas in [4.29] the converse choice is made.

Since Y_0 is periodic with respect to t^+, the curves in the Y_0, $(\partial Y_0/\partial t^+)$-plane for fixed $\tilde{t}$ (hence with E_0 and ω_0 also fixed) are ovals that are symmetric with respect to the Y_0 axis as sketched in Figure 4.4.1. Note that since $\tilde{t}$ is held fixed, the closed curve in Figure 4.4.1 is not an actual integral curve of (4.4.8a). However, we expect E_0, ω_0 and $\tilde{t}$ to change only by $O(\epsilon)$ after one complete cycle in this "phase-plane".

Given initial values for y and $\dot{y}$ at $t = 0$ we have, once $\omega_0(E_0)$ is determined, the initial values $Y_0(\phi_0(0), 0)$ and $(\partial Y_0(\phi_0(0), 0)/\partial t^+)$, which specify a point, say the point marked O, on the oval in Figure 4.4.1. The details of this calculation are discussed at the end of this section.

Figure 4.4.1 "Phase-Plane" of Y_0, $\frac{\partial Y_0}{\partial t^+}$ for Fixed E_0, ω_0 and $\tilde{t}$

With no loss of generality we choose $\phi_0(0)$ so that $t^+ = 0$ when Y_0 first equals $Y_{0_{min}}$. Note $\phi_0(0) < 0$. Thus, $Y_0(0, -\epsilon\phi_0(0)) = Y_{0_{min}}$, $(\partial Y_0(0, -\epsilon\phi_0(0))/\partial t^+) = 0$, and Y_0 is an even function of t^+. Moreover, the expression for $f(t^+, E_0, \omega_0, \tilde{t})$ is obtained by inversion from (Note that $\lambda_0 \equiv 0$)

$$t^+ = \omega_0(\tilde{t}) \int_{Y_{0_{min}}(E_0, \tilde{t})}^{Y_0(t^+, \tilde{t})} \frac{d\eta}{\pm\sqrt{2[E_0(\tilde{t}) - V(\eta, \tilde{t})]}}, \qquad (4.4.10)$$

where the $\pm$ signs correspond to the signs of $\partial Y_0/\partial t^+$.

The period of oscillation P is then twice the integral from $Y_{0_{min}}$ to $Y_{0_{max}}$, i.e.,

$$P(E_0, \omega_0, \tilde{t}) \equiv 2\omega_0 \int_{Y_{0_{min}}(E_0, \tilde{t})}^{Y_{0_{max}}(E_0, \tilde{t})} \frac{dY_0}{\sqrt{2[E_0(\tilde{t}) - V(Y_0, \tilde{t})]}}. \qquad (4.4.11)$$

At this stage, P is a function of E_0, ω_0 and $\tilde{t}$; once $E_0(\tilde{t})$ and $\omega_0(\tilde{t})$ are defined we will have P as a function of $\tilde{t}$. In [4.29] Luke points out that unless P is a *constant* we lose uniformity for t large. To see this, let us use construct the general solution for Y_1. Unfortunately, this form of Y_1 is not convenient for the calculation of the equation governing ϕ_0. We will follow the approach used by Bourland and Haberman in [4.30].

In order to construct a second linearly independent solution of $L(Y_1) = 0$ depending on t^+, E_0 and $\tilde{t}$ we begin by writing (4.4.7a) in the form

$$\Omega^2(E_0, \tilde{t}) \frac{\partial^2 p}{\partial t^{+^2}}(t^+, E_0, \tilde{t}) + g(p(t^+, E_0, \tilde{t}), \tilde{t}) = 0, \tag{4.4.18}$$

where we have used (4.4.15) to express ω_0 in terms of E_0 and $\tilde{t}$. Now, taking the partial derivative of (4.4.18) with respect to E_0, holding t^+ and $\tilde{t}$ fixed, gives

$$2\Omega \frac{\partial \Omega}{\partial E_0} \frac{\partial^2 p}{\partial t^{+^2}} + \Omega^2 \frac{\partial^2}{\partial t^{+^2}} \left(\frac{\partial p}{\partial E_0} \right) + g_y(p, \tilde{t}) \frac{\partial p}{\partial E_0} = 0,$$

i.e.,

$$L\left(\frac{\partial p}{\partial E_0} \right) = -2\Omega \frac{\partial \Omega}{\partial E_0} \frac{\partial^2 p}{\partial t^{+^2}}. \tag{4.4.19}$$

Next we compute $L(t^+ \frac{\partial p}{\partial t^+})$; we have

$$\begin{aligned} L\left(t^+ \frac{\partial p}{\partial t^+}\right) &= \Omega^2 \frac{\partial^2}{\partial t^{+^2}} \left(t^+ \frac{\partial p}{\partial t^+}\right) + g_y(p, \tilde{t}) t^+ \frac{\partial p}{\partial t^+} \\ &= 2\Omega^2 \frac{\partial^2 p}{\partial t^{+^2}} + t^+ L\left(\frac{\partial p}{\partial t^+}\right) = 2\Omega^2 \frac{\partial^2 p}{\partial t^{+^2}} \end{aligned} \tag{4.4.20}$$

because $L(\partial p / \partial t^+) = 0$. Using (4.4.19) and (4.4.20) we see that

$$L\left(\Omega \frac{\partial p}{\partial E_0} + \frac{\partial \Omega}{\partial E_0} t^+ \frac{\partial p}{\partial t^+} \right) = \Omega L\left(\frac{\partial p}{\partial \tilde{t}_0} \right) + \frac{\partial \Omega}{\partial E_0} L\left(t^+ \frac{\partial p}{\partial t^+} \right) = 0.$$

Therefore,

$$q(t^+, E_0, \tilde{t}) \equiv \Omega(E_0, \tilde{t}) \frac{\partial p}{\partial E_0}(t^+, E_0, \tilde{t}) + \frac{\partial \Omega}{\partial E_0}(E_0, \tilde{t}) t^+ \frac{\partial p}{\partial t^+}(t^+, E_0, \tilde{t}) \tag{4.4.21}$$

is a second homogeneous solution ($L(q) = 0$) that is even in t^+. To ascertain that $(\partial p / \partial t^+)$ and q are linearly independent we construct the Wronskian

$$\begin{aligned} W &= \frac{\partial p}{\partial t^+} \frac{\partial q}{\partial t^+} - q \frac{\partial^2 p}{\partial t^{+^2}} \\ &= \frac{\partial p}{\partial t^+} \left(\Omega \frac{\partial^2 p}{\partial E_0 \partial t^+} + \frac{\partial \Omega}{\partial E_0} \frac{\partial p}{\partial t^+} + \frac{\partial \Omega}{\partial E_0} t^+ \frac{\partial^2 p}{\partial t^{+^2}} \right) \\ &\quad - \frac{\partial^2 p}{\partial t^{+^2}} \left(\Omega \frac{\partial p}{\partial E_0} + \frac{\partial \Omega}{\partial E_0} t^+ \frac{\partial p}{\partial t^+} \right). \end{aligned} \tag{4.4.22}$$

Note that in evaluating $r_{1_{odd}}$ we have used $\omega_0(\tilde{t}) = \Omega(E_0(\tilde{t}), \tilde{t})$ and $Y_0(t^+, \tilde{t}) = p(t^+, E_0(\tilde{t}), \tilde{t})$. In particular, the bracketed term on the right-hand side of (4.4.28b) is just $(\partial^2 Y_0/\partial t^+ \partial\tilde{t})$.

The particular solution of (4.4.7b) due to $r_{1_{even}}$ follows from (4.4.20) in the form

$$y_{1p_{even}}(t^+, E_0, \tilde{t}) = -\frac{\omega_1}{\Omega} t^+ \frac{\partial p}{\partial t^+}, \tag{4.4.29a}$$

and we use variation of parameters to compute the odd particular solution in the form

$$y_{1p_{odd}}(t^+, E_0, \tilde{t}) = \frac{q(t^+, E_0, \tilde{t})}{\Omega(E_0, \tilde{t})} \int_0^{t^+} r_{1_{odd}}(s, E_0, \tilde{t}) \frac{\partial p}{\partial t^+}(s, E_0, \tilde{t}) ds$$
$$- \frac{1}{\Omega(E_0, \tilde{t})} \frac{\partial p}{\partial t^+}(t^+, E_0, \tilde{t}) \int_0^{t^+} r_{1_{odd}}(s, E_0, \tilde{t}) q(s, E_0, \tilde{t}) ds. \tag{4.4.29b}$$

The general solution for Y_1 then has the form

$$Y_1(t^+, \tilde{t}) \equiv y_1(t^+, E_0, \tilde{t}) = A_1(\tilde{t}) \frac{\partial p}{\partial t^+}(t^+, E_0, \tilde{t}) + B_1(\tilde{t}) q(t^+, E_0, \tilde{t})$$
$$+ y_{1p_{odd}}(t^+, E_0, \tilde{t}) + y_{1p_{even}}(t^+, E_0, \tilde{t}). \tag{4.4.30}$$

In order that Y_1 be a periodic function of t^+, its even and odd parts must *individually* be periodic. Consider first the even part of Y_1, i.e., $B_1 q + y_{1p_{even}}$. Using (4.4.21) for q, (4.4.29a) for $y_{1p_{even}}$ and noting that $(\partial p/\partial t^+)$ is periodic in t^+, we see that the mixed secular terms are eliminated by setting

$$B_1(\tilde{t}) \frac{\partial \Omega}{\partial E_0}(E_0, \tilde{t}) - \frac{\omega_1(\tilde{t})}{\Omega(E_0, \tilde{t})} = 0, \tag{4.4.31}$$

and (4.4.30) reduces to

$$Y_1(t^+, \tilde{t}) \equiv y_1(t^+, E_0, \tilde{t}) = A_1(\tilde{t}) \frac{\partial p}{\partial t^+}(t^+, E_0, \tilde{t})$$
$$+ \frac{\omega_1}{(\partial\Omega/\partial E_0)} \frac{\partial p}{\partial E_0}(t^+, E_0, \tilde{t}) + y_{1p_{odd}}. \tag{4.4.32}$$

If $(\partial\Omega/\partial E_0) \neq 0$, (4.4.31) determines B_1 once E_0 and ϕ_0 have been calculated (Note: $\omega_1 = (d\phi_0/d\tilde{t})$). Thus, (4.4.31) makes no contribution towards the determination of the two unknowns (E_0, ϕ_0) in the $O(1)$ solution unless $(\partial\Omega/\partial E_0) = 0$, in which case we must set $\omega_1 = 0$.

The odd part of y_1 consists of $A_1(\partial p/\partial t^+) + y_{1p_{odd}}$, and since $(\partial p/\partial t^+)$ is already periodic, we must require $y_{1p_{odd}}$ to be periodic by itself. We show next that a necessary and sufficient condition for $y_{1p_{odd}}$ to be periodic is that

$$\int_0^{P_0} r_{1_{odd}}(t^+, E_0, \tilde{t}) \frac{\partial p}{\partial t^+}(t^+, E_0, \tilde{t}) dt^+ = 0. \tag{4.4.33}$$

In general this is a nonlinear first-order equation for E_0 and its solution defines $E_0(\tilde{t})$ for a given $E_0(0)$.

If h is linear in $\dot{y}$, and does not depend on y, i.e., $h = \tilde{h}(\tilde{t})\dot{y}$, (4.4.37) simplifies further to the linear equation

$$\frac{dJ}{d\tilde{t}} + \tilde{h}(\tilde{t})J = 0. \tag{4.4.38}$$

This has the solution

$$J(E_0, \tilde{t}) = J(E_0(0), 0)\exp(-\int_0^{\tilde{t}} \tilde{h}(s)ds). \tag{4.4.39}$$

Inverting (4.4.36b) then gives $E_0(\tilde{t})$.

If there is no dissipation ($h = 0$) we find J = constant. This is the generalization (for a nonlinear oscillator) of the adiabatic invariant we computed in Sec. 4.3.2 for the linear oscillator with slowly varying frequency.

Weakly nonlinear problem: $g = \mu^2(\tilde{t})y$

It is instructive to specialize the foregoing results to the case where g is linear in y, i.e., $g = \mu^2(\tilde{t})y$ with $\mu(\tilde{t}) \neq 0$ prescribed. For the time being we leave h in its general form.

Equation (4.4.7a) becomes

$$\omega_0^2 \frac{\partial^2 Y_0}{\partial t^{+2}} + \mu^2 Y_0 = 0 \tag{4.4.40}$$

with solution

$$f(t^+, E_0, \omega_0, \tilde{t}) = \frac{\sqrt{2E_0}}{\mu} \cos \frac{\mu}{\omega_0} t^+. \tag{4.4.41}$$

Thus, $Y_{0_{max}} = \sqrt{2E_0}/\mu$ and $Y_{0_{min}} = -\sqrt{2E_0}/\mu$. If we choose $P_0 = 2\pi$, the equation (4.4.11) for the period becomes

$$\pi = \omega_0 \int_{-\sqrt{2E_0}/\mu}^{\sqrt{2E_0}/\mu} \frac{dY_0}{\sqrt{2E_0 - Y_0^2\mu^2}} = \frac{\omega_0}{\mu}\pi, \tag{4.4.42}$$

and this gives $\omega_0 = \mu$. Thus, (4.4.41) implies

$$p(t^+, E_0, \tilde{t}) = \frac{\sqrt{2E_0}}{\mu} \cos t^+. \tag{4.4.43}$$

We note that the two linearly independent solutions of $L(Y_1) = 0$, i.e., $(\partial p/\partial t^+) = -(\sqrt{2E_0}/\mu)\sin t^+$ and $q = \mu(\partial p/\partial E_0) = (1/\sqrt{2E_0})\cos t^+$, are both periodic in this case.

Since $(\partial\Omega/\partial E_0) = 0$, (4.4.31) gives $\omega_1(\tilde{t}) = 0$, i.e., $\phi_0(\tilde{t}) = \phi_0(0) =$ constant. The second periodicity condition (4.4.35) gives

$$\frac{d}{d\tilde{t}}\left(\frac{E_0}{\mu}\right) - \frac{1}{2\pi}\int_0^{2\pi} h\left(\frac{2E_0}{\mu}\cos s, -\sqrt{2E_0}\sin s, \tilde{t}\right)\frac{\sqrt{2E_0}}{\mu}\sin s ds = 0. \tag{4.4.44}$$

derived more directly. This derivation is based on the observation made in [4.32] that (4.4.1) implies an *exact* condition for the action.

If we regard $y = Y(t^+, \tilde{t}; \epsilon)$ with $\tilde{t} = \epsilon t$ and t^+ to be defined by (4.4.4) exactly, we find that (4.4.1) becomes

$$(\omega_0 + \epsilon\omega_1)^2 \frac{\partial^2 Y}{\partial t^{+2}} + g(Y, \tilde{t}) + \epsilon \left[\left(\frac{d\omega_0}{d\tilde{t}} + \epsilon \frac{d\omega_1}{d\tilde{t}} \right) \frac{\partial Y}{\partial t^+} \right.$$

$$\left. +2(\omega_0 + \epsilon\omega_1) \frac{\partial^2 Y}{\partial t^+ \partial \tilde{t}} + h(Y, (\omega_0 + \epsilon\omega_1) \frac{\partial Y}{\partial t^+} + \epsilon \frac{\partial Y}{\partial \tilde{t}}, \tilde{t}) \right] + \epsilon^2 \frac{\partial^2 Y}{\partial \tilde{t}^2} = 0, \tag{4.4.49}$$

also exactly.

Let us now multiply (4.4.49) by $(\partial Y / \partial t^+)$ and integrate the result with respect to t^+ from $t^+ = 0$ to $t^+ = P_0$

$$(\omega_0 + \epsilon\omega_1)^2 \int_0^{P_0} \frac{\partial^2 Y}{\partial t^{+2}} \frac{\partial Y}{\partial t^+} dt^+ + \int_0^{P_0} g(Y, \tilde{t}) \frac{\partial Y}{\partial t^+} dt^+$$

$$+\epsilon \left(\frac{d\omega_0}{d\tilde{t}} + \epsilon \frac{d\omega_1}{d\tilde{t}} \right) \int_0^{P_0} \left(\frac{\partial Y}{\partial t^+} \right)^2 dt^+ + 2\epsilon(\omega_0 + \epsilon\omega_1) \int_0^{P_0} \frac{\partial^2 Y}{\partial t^+ \partial \tilde{t}} \frac{\partial Y}{\partial t^+} dt^+$$

$$+\epsilon \int_0^{P_0} h(Y, (\omega_0 + \epsilon\omega_1) \frac{\partial Y}{\partial t^+} + \epsilon \frac{\partial Y}{\partial \tilde{t}}, \tilde{t}) \frac{\partial Y}{\partial t^+} dt^+ + \epsilon^2 \int_0^{P_0} \frac{\partial^2 Y}{\partial \tilde{t}^2} \frac{\partial Y}{\partial t^+} dt^+ = 0.$$

The first two terms are

$$\frac{1}{2} (\omega_0 + \epsilon\omega_1)^2 \int_0^{P_0} \frac{\partial}{\partial t^+} \left(\frac{\partial Y}{\partial t^+} \right)^2 dt^+, \quad \int_0^{P_0} \frac{\partial}{\partial t^+} (V(Y, \tilde{t})) dt^+,$$

and vanish because Y is periodic in t^+ with period P_0. The third and fourth terms combine and upon dividing out an ϵ we find

$$\frac{d}{d\tilde{t}} \left[(\omega_0 + \epsilon\omega_1) \int_0^{P_0} \left(\frac{\partial Y}{\partial t^+} \right)^2 dt^+ \right]$$

$$+ \int_0^{P_0} h(Y, (\omega_0 + \epsilon\omega_1) \frac{\partial Y}{\partial t^+} + \epsilon \frac{\partial Y}{\partial \tilde{t}}, \tilde{t}) \frac{\partial Y}{\partial t^+} dt^+$$

$$+\epsilon \int_0^{P_0} \frac{\partial^2 Y}{\partial \tilde{t}^2} \frac{\partial Y}{\partial t^+} dt^+ = 0. \tag{4.4.50}$$

For a solution of (4.4.4) that is periodic in t^+ with period P_0, (4.4.50) is an *exact* result.

Now if we expand Y as in (4.4.3), the $O(1)$ and $O(\epsilon)$ terms of the expansion of (4.4.50) satisfy

$$\frac{d}{d\tilde{t}} \left[\omega_0 \int_0^{P_0} \left(\frac{\partial Y_0}{\partial t^+} \right)^2 dt^+ \right] + \int_0^{P_0} h \frac{\partial Y_0}{\partial t^+} dt^+ = 0 \tag{4.4.51}$$

$$\omega_0 \int_0^{P_0} \frac{\partial h}{\partial \dot{y}} \frac{\partial Y_1}{\partial t^+} \frac{\partial Y_0}{\partial t^+} dt^+ = \omega_0 \int_0^{P_0} \frac{\partial h}{\partial \dot{y}} \frac{\partial y_{1_{\text{even}}}}{\partial t^+} \frac{\partial Y_0}{\partial t^+} dt^+$$

$$= \frac{\Omega\omega_1}{\partial\Omega/\partial E_0} \int_0^{P_0} \frac{\partial h}{\partial \dot{y}}(p, \Omega\frac{\partial p}{\partial t^+}, \tilde{t}) \frac{\partial^2 p}{\partial E_0 \partial t^+} \frac{\partial p}{\partial t^+} dt^+ \qquad (4.4.53c)$$

$$\int_0^{P_0} \frac{\partial h}{\partial \dot{y}} \frac{\partial Y_0}{\partial \tilde{t}} \frac{\partial Y_0}{\partial t^+} dt^+ = 0 \qquad (4.4.53d)$$

$$\int_0^{P_0} h \frac{\partial Y_1}{\partial t^+} dt^+ = \int_0^{P_0} h \frac{\partial y_{1_{\text{even}}}}{\partial t^+} dt^+$$

$$= \frac{\omega_1}{\partial\Omega/\partial E_0} \int_0^{P_0} h(p, \Omega\frac{\partial p}{\partial t^+}, \tilde{t}) \frac{\partial^2 p}{\partial E_0 \partial t^+} dt^+. \qquad (4.4.53e)$$

Finally, since $(\partial^2 Y_0/\partial t^+)$ is even and $(\partial Y_0/\partial t^+)$ is odd, the last term in (4.4.52) vanishes. When the expressions in (4.4.53) are used to simplify (4.4.52) we find

$$\frac{d}{d\tilde{t}} \left[\frac{2\Omega\omega_1}{\partial\Omega/\partial E_0} \int_0^{P_0} \frac{\partial p}{\partial t^+} \frac{\partial^2 p}{\partial E_0 \partial t^+} dt^+ + \omega_1 \int_0^{P_0} \left(\frac{\partial p}{\partial t^+}\right)^2 dt^+ \right]$$

$$+ \frac{\omega_1}{\partial\Omega/\partial E_0} \int_0^{P_0} \left\{ h(p, \Omega\frac{\partial p}{\partial t^+}, \tilde{t}) \frac{\partial^2 p}{\partial E_0 \partial t^+} + \frac{\partial h}{\partial y}(p, \Omega\frac{\partial p}{\partial t^+}, \tilde{t}) \frac{\partial p}{\partial E_0} \frac{\partial p}{\partial t^+} \right.$$

$$\left. + \frac{\partial h}{\partial \dot{y}}(p, \Omega\frac{\partial p}{\partial t^+}, \tilde{t}) \frac{\partial p}{\partial t^+} \left[\Omega \frac{\partial^2 p}{\partial E_0 \partial t^+} + \frac{\partial \Omega}{\partial E_0} \left(\frac{\partial p}{\partial t^+}\right)^2 \right] \right\} dt^+ = 0. \qquad (4.4.54)$$

This is just the linear homogeneous equation

$$\frac{d}{d\tilde{t}} \left[\frac{\omega_1}{\partial\Omega/\partial E_0} \frac{\partial}{\partial E_0} \left(\Omega \int_0^{P_0} \left(\frac{\partial p}{\partial t^+}\right)^2 dt^+ \right) \right]$$

$$+ \frac{\omega_1}{\partial\Omega/\partial E_0} \frac{\partial}{\partial E_0} \int_0^{P_0} h(p, \Omega\frac{\partial p}{\partial t^+}, \tilde{t}) \frac{\partial p}{\partial t^+} dt^+ = 0. \qquad (4.4.55)$$

Using the notation (4.4.36) for the action J and dissipation D gives

$$\frac{d}{d\tilde{t}} \left(\frac{J_{E_0}}{\Omega_{E_0}} \omega_1 \right) + \frac{D_{E_0}}{\Omega_{E_0}} \omega_1 = 0, \qquad (4.4.56)$$

where $\Omega_{E_0} \equiv (\partial\Omega/\partial E_0)$, $J_{E_0} \equiv (\partial J/\partial E_0)$ and $D_{E_0} \equiv (\partial D/\partial E_0)$. We can compute Ω_{E_0} using the expression (4.4.15)

$$\Omega_{E_0} = -\frac{P_0}{2} \left\{ \int_{Y_{0_{\min}}(E_0,\tilde{t})}^{Y_{0_{\max}}(E_0,\tilde{t})} \frac{dY_0}{\sqrt{2[E_0(\tilde{t}) - V(Y_0, \tilde{t})]}} \right\}^{-2}$$

where α_0, α_1, β_0 and β_1 are specified constants. Since $t^+ = \phi_0(0)$ and $\tilde{t} = 0$ at $t = 0$, (4.4.60a) and the expansion (4.4.3) for y give

$$Y_0(\phi_0(0), 0) = \alpha_0 \tag{4.4.61a}$$

$$Y_1(\phi_0(0), 0) = \alpha_1. \tag{4.4.61b}$$

Similarly, using the expansion (4.4.5a) for $\dot{y}$, we see that (4.4.60b) gives

$$\omega_0(0)\frac{\partial Y_0}{\partial t^+}(\phi_0(0), 0) = \beta_0 \tag{4.4.62a}$$

$$\omega_0(0)\frac{\partial Y_1}{\partial t^+}(\phi_0(0), 0) + \omega_1(0)\frac{\partial Y_0}{\partial t^+}(\phi_0(0), 0) + \frac{\partial Y_0}{\partial \tilde{t}}(\phi_0(0), 0) = \beta_1. \tag{4.4.62b}$$

If we use the definition (4.4.16) for p, the initial condition (4.4.61a) for Y_0 becomes

$$p(\phi_0(0), E_0(0), 0) = \alpha_0. \tag{4.4.63a}$$

Similarly, (4.4.62a) has the form

$$\Omega(E_0(0), 0)\frac{\partial p}{\partial t^+}(\phi_0(0), E_0(0), 0) = \beta_0. \tag{4.4.63b}$$

These two algebraic equations define the two unknowns $\phi_0(0)$ and $E_0(0)$ in terms of the specified constants α_0, β_0. With $E_0(0)$ known, the solution of (4.4.37) defines $E_0(\tilde{t})$. Using this $E_0(\tilde{t})$ in (4.4.15) for $\Omega(E_0(\tilde{t}), \tilde{t})$ and in (4.4.16) for $p(t^+, E_0(\tilde{t}), \tilde{t})$ specifies $\omega_0(\tilde{t})$ and $Y_0(t^+, \tilde{t})$ completely. In order to complete the solution to $O(1)$ we need to know $\omega_1(0)$ in order to specify $\omega_1(\tilde{t})$ from (4.4.59). To evaluate $\omega_1(0)$ we consider the initial conditions to $O(\epsilon)$.

Using the now known expression for $Y_0(t^+, \tilde{t})$, and using (4.4.32) for Y_1 in (4.4.61b) and (4.4.62b) gives the following pair of linear algebraic equations for $A_1(0)$ and $\omega_1(0)$

$$A_1(0)\frac{\partial Y_0}{\partial t^+}(\phi_0(0), 0) + \frac{\omega_1(0)}{\Omega_{E_0}(E_0(0), 0)}\frac{\partial p}{\partial E_0}(\phi_0(0), E_0(0), 0)$$
$$= \alpha_1 - y_{1p_{odd}}(\phi_0(0), E_0(0), 0) \tag{4.4.64a}$$

$$A_1(0)\omega_0(0)\frac{\partial^2 Y_0}{\partial t^{+^2}}(\phi_0(0), 0) + \frac{\omega_1(0)}{\Omega_{E_0}(E_0(0), 0)}\left[\omega_0(0)\frac{\partial^2 p}{\partial E_0 \partial t^+}(\phi_0(0), E_0, 0)\right.$$
$$\left. + \Omega_{E_0}(E_0(0), 0)\frac{\partial Y_0}{\partial t^+}(\phi_0(0), 0)\right] = \beta_1 - \frac{\partial Y_0}{\partial \tilde{t}}(\phi_0(0), 0)$$
$$-\omega_0(0)\frac{\partial y_{1p_{odd}}}{\partial t^+}(\phi_0(0), E_0(0), 0). \tag{4.4.64b}$$

and find

$$\omega_0^2 \left(\frac{\partial^2 Y_0}{\partial t^{+2}} y_{1p_{odd}} - \frac{\partial Y_0}{\partial t^+} \frac{\partial y_{1p_{odd}}}{\partial t^+} \right)_{t^+=\phi_0(0),\tilde{t}=0} =$$

$$\left\{ \frac{\partial}{\partial \tilde{t}} \left[\omega_0(\tilde{t}) \int_0^{t^+} \left(\frac{\partial Y_0}{\partial t^+}(s,\tilde{t}) \right)^2 ds \right] \right.$$

$$\left. + \int_0^{t^+} \frac{\partial Y_0}{\partial t^+}(s,\tilde{t}) h(Y_0(s,\tilde{t}), \frac{\partial Y_0}{\partial t^+}(s,\tilde{t}), \tilde{t}) ds \right\}_{t^+=\phi_0(0),\tilde{t}=0} .$$

Therefore, (4.4.65) has the explicit form

$$\begin{aligned} \frac{\omega_1(0)}{\Omega_{E_0}(E_0(0),0)} = & \left\{ \frac{\partial}{\partial \tilde{t}} \left[\omega_0(\tilde{t}) \int_0^{t^+} \left(\frac{\partial Y_0}{\partial t^+}(s,\tilde{t}) \right)^2 ds \right] \right. \\ & + \int_0^{t^+} \frac{\partial Y_0}{\partial t^+}(s,\tilde{t}) h(Y_0(s,\tilde{t}), \frac{\partial Y_0}{\partial t^+}(s,\tilde{t}), \tilde{t}) ds \\ & - \omega_0(\tilde{t}) \frac{\partial Y_0}{\partial t^+}(t^+,\tilde{t}) \frac{\partial Y_0}{\partial \tilde{t}}(t^+,\tilde{t}) + \omega_0(\tilde{t}) \left[\beta_1 \frac{\partial Y_0}{\partial t^+}(t^+,\tilde{t}) \right. \\ & \left. \left. - \omega_0(\tilde{t}) \alpha_1 \frac{\partial^2 Y_0}{\partial t^{+2}}(t^+,\tilde{t}) \right] \right\}_{t^+=\phi_0(0),\tilde{t}=0} . \end{aligned} \quad (4.4.66)$$

As pointed out in [4.30], this expression simplifies further if we use the energy equation (4.4.8a). Taking the total derivative of this expression with respect to $\tilde{t}$ gives

$$\omega_0 \omega_0' \left(\frac{\partial Y_0}{\partial t^+} \right)^2 + \omega_0^2 \frac{\partial Y_0}{\partial t^+} \frac{\partial^2 Y_0}{\partial t^+ \partial \tilde{t}} + \frac{\partial V}{\partial y}(Y_0,\tilde{t}) \frac{\partial Y_0}{\partial \tilde{t}} + \frac{\partial V}{\partial \tilde{t}}(Y_0,\tilde{t}) = E_0' \quad (4.4.67)$$

where $' = d/d\tilde{t}$. We now use (4.4.7a) to set $V_y = g = -\omega_0^2(\partial^2 Y_0/\partial t^{+2})$, divide out an ω_0 and write the result as

$$\omega_0' \left(\frac{\partial Y_0}{\partial t^+} \right)^2 + 2\omega_0 \frac{\partial Y_0}{\partial t^+} \frac{\partial^2 Y_0}{\partial t^+ \partial \tilde{t}} - \omega_0 \left(\frac{\partial Y_0}{\partial t^+} \frac{\partial^2 Y_0}{\partial t^+ \partial \tilde{t}} + \frac{\partial Y_0}{\partial \tilde{t}} \frac{\partial^2 Y_0}{\partial t^{+2}} \right)$$

$$= \frac{1}{\omega_0} E_0' - \frac{1}{\omega_0} \frac{\partial V}{\partial \tilde{t}} .$$

But, this is

$$\frac{\partial}{\partial \tilde{t}} \left[\omega_0 \left(\frac{\partial Y_0}{\partial t^+} \right)^2 \right] - \omega_0 \frac{\partial}{\partial t^+} \left(\frac{\partial Y_0}{\partial t^+} \frac{\partial Y_0}{\partial \tilde{t}} \right) = \frac{1}{\omega_0} E_0' - \frac{1}{\omega_0} \frac{\partial V}{\partial \tilde{t}} .$$

$$- \frac{1}{\omega_0(0)} \frac{\partial V}{\partial \tilde{t}} (Y_0(s,0),0) \Big] \, ds \tag{4.4.72a}$$

$$C_2(\alpha_0, \beta_0) = -\omega_0^2(0) \frac{\partial^2 Y_0}{\partial t^{+2}} (\phi_0(0), 0) \tag{4.4.72b}$$

$$C_3(\alpha_0, \beta_0) = \omega_0(0) \frac{\partial Y_0}{\partial t^+} (\phi_0(0), 0). \tag{4.4.72c}$$

As pointed out earlier, and indicated by the arguments of C_1, C_2 and C_3, these constants involve functions that are completely defined once α_0 and β_0 are specified. Therefore, if α_1 and β_1 are also prescribed arbitrarily, (4.4.71) shows that $\omega_1(0)$ does not vanish in general; it only vanishes for the one parameter family of values of α_1, β_1, for which

$$C_1 + C_2\alpha_1 + C_3\beta_1 = 0. \tag{4.4.73}$$

With $\omega_1(0) \neq 0$, (4.4.59) gives $\omega_1(\tilde{t}) \neq 0$, i.e., the phase-shift of the $O(1)$ oscillations is not constant. A special case for which $\omega_1 \equiv 0$ corresponds to $h \equiv 0$, $V_{\tilde{t}} = 0$ and $\alpha_1 = \beta_1 = 0$.

In [4.31], initial conditions for which (4.4.73) holds, and hence $\omega_1 \equiv 0$, are denoted as 'synchronized' initial conditions as the solution is significantly simplified. It is also pointed out there that for any *numerically* prescribed set of values for ϵ, $y(0;\epsilon)$ and $\dot{y}(0;\epsilon)$, it is always possible to choose α_0, β_0, α_1 and β_1 consistent with the initial data and such that (4.4.73) is satisfied. Thus, any pair of initial values $y(0;\epsilon)$ and $\dot{y}(0;\epsilon)$ can be regarded as synchronized, and we need henceforth not dwell on the variation of the phase-shift.

4.4.2 An Example

We consider the problem discussed in [4.6] and generalize this to include a small damping term that is linear in $\dot{y}$ and slowly varying

$$\ddot{y} + \epsilon\tilde{h}(\tilde{t})\dot{y} + a(\tilde{t})y + b(\tilde{t})y^3 = 0 \tag{4.4.74}$$

where $\tilde{h}$, a and b are given functions.

The energy integral (4.4.8a) is

$$\frac{\omega_0^2}{2} \left(\frac{\partial Y_0}{\partial t^+} \right)^2 + V(Y_0, \tilde{t}) = E_0(\tilde{t}) \tag{4.4.75}$$

where

$$V(Y_0, \tilde{t}) = \frac{a(\tilde{t})}{2} Y_0^2 + \frac{b(\tilde{t})}{4} Y_0^4. \tag{4.4.76}$$

Examining V for the different possible combinations of the signs of a and b will determine the cases for which (4.4.75) admits periodic solutions.

It is convenient to introduce the notation

$$\nu \equiv -b(\tilde{t})Y_{0_{\max}}^4/4E_0 \tag{4.4.80a}$$

which implies (using the expression given by (4.4.77) for $Y_{0_{\max}}^4$)

$$1+\nu = 1 - \frac{b}{4E_0}\left[\frac{4E_0}{b} - 2\frac{a}{b}Y_{0_{\max}}^2\right] = \frac{a}{2E_0}Y_{0_{\max}}^2. \tag{4.4.80b}$$

Thus, (4.4.79) takes the form

$$t^+ = \frac{\omega_0(\tilde{t})}{\sqrt{2E_0}}Y_{0_{\max}}\int_{-1}^{Y_0/Y_{0_{\max}}} \frac{d\xi}{\sqrt{(1-\xi^2)(1-\nu\xi^2)}}. \tag{4.4.81}$$

Let us first apply the condition (4.4.11) that ensures the period is independent of $\tilde{t}$; here it is convenient to choose $P_0 = 4$. We find

$$4 = 2\omega_0 \int_{-A_0}^{A_0} \frac{dY_0}{\sqrt{2[E_0 - a(\tilde{t})Y_0^2/2 - b(\tilde{t})Y_0^4/4]}}. \tag{4.4.82}$$

where we have set $Y_{0_{\max}} = A_0$. Notice that (4.4.78) gives a relation linking A_0 to E_0 and the two known functions $a(\tilde{t})$, $b(\tilde{t})$. Introducing the change of variable $\xi = Y_0/A_0$ as in (4.4.79) we find

$$4 = \frac{2\omega_0 A_0}{\sqrt{2E_0}}\int_{-1}^{1}\frac{d\xi}{\sqrt{(1-\xi^2)(1-\nu\xi^2)}}. \tag{4.4.83}$$

The integral in (4.4.83) is just $2K(\nu)$, where K is the complete elliptic integral of the first kind (cf. (4.19))

$$K(\nu) = \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\nu\xi^2)}}. \tag{4.4.84}$$

Therefore, (4.4.83) reduces to

$$\omega_0 = \frac{\sqrt{2E_0}}{K(\nu)A_0}. \tag{4.4.85}$$

It follows from the definition of ν and A_0 that the right-hand side of (4.4.85) is a known function of E_0 and the given functions $a(\tilde{t})$ and $b(\tilde{t})$. Thus, (4.4.85) defines $\Omega(E_0, \tilde{t})$ of (4.4.15). Henceforth, we will use A_0 instead of E_0 in our calculations.

In preparation for inverting (4.4.81) we isolate the integral over $(-1, 0)$ and use (4.4.84)-(4.4.85) to obtain

$$t^+ = 1 + \frac{1}{K(\nu)}\int_0^{Y_0/A_0}\frac{d\xi}{\sqrt{(1-\xi^2)(1-\nu\xi^2)}}. \tag{4.4.86}$$

The inverse is then expressed in terms of the elliptic sine function in the form:

$$Y_0 = A_0(\tilde{t})\,\mathrm{sn}\left[K(\nu)(t^+ - 1), \nu\right]. \tag{4.4.87}$$