Asymptotic approximations of oscillations

There are several kinds of problems

Second order eqns

$$y'' + F(y, y', t, \varepsilon) = 0 \qquad y(0) = a \quad y'(0) = b$$

Systems

$$y' = F(y, t, \varepsilon) \qquad y(0) = a \qquad y \in \mathbb{R}^n$$

Systems in Standard Form

$$y' = \omega + \varepsilon F(y, t, \varepsilon) \qquad y(0) = a \qquad y \in \mathbb{R}^n$$

Perturbed Hamiltonian Systems $H = H(p, q, t, \varepsilon)$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + \varepsilon G_i(q, p, t)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + \varepsilon H_i(q, p, t)$$

for $i = 1, \dots n$.

If for $F|_{\varepsilon=0}$ the system is linear the system is often said to be <u>weakly nonlinear</u>.

If the system is Hamiltonian for $G_i = H_i = 0$ it is said to be <u>nearly Hamiltonian</u>.

Regular expansions such as

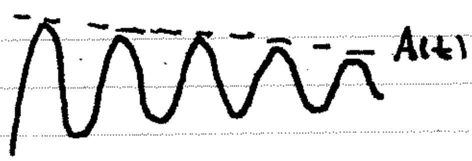
$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2)$$

yield approximations valid for $t \in [0, T_1]$, $T = O(1)$.

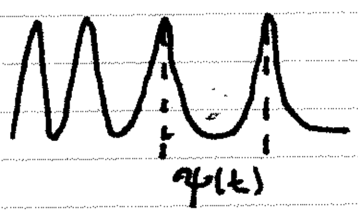
Such solutions do not account for cumulative effects for long times such as

$$t = O\left(\frac{1}{\varepsilon^n}\right) \qquad n \geq 1$$

Two cumulative effects are



slowly varying amplitude $A(t)$



slowly varying phase $\psi(t)$

Broadly there are two goals

a) Find approximations $Y_0(t, \tilde{t})$, $\tilde{t} = \varepsilon t$ slowtime that capture cumulative effects s.t

$$y(t,\varepsilon) - Y_0(t,\tilde{t}) = O(\varepsilon) \qquad t = O(\varepsilon^{-1})$$

b) Identify (stable) periodic attractors such as limit cycles and dynamics on tori.

EXAMPLE **Failure in regular expansions**

$$\boxed{\begin{array}{l} y'' + \varepsilon y' + y = 0 \\ y(0) = 0 \qquad y'(0) = 1 \end{array}}$$

Exact solution

$$y(t,\varepsilon) = \frac{1}{\sqrt{1-\varepsilon^2/4}} e^{-\varepsilon t/2} \sin\left(t\sqrt{1-\varepsilon^2/4}\right)$$

Regular expansion

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2)$$

yields

$O(1)$ $\quad y_0'' + y_0 = 0 \qquad y_0(0) = 0 \quad y_0'(0) = 1$

$O(\varepsilon)$ $\quad y_1'' + y_1 = -y_0' \qquad y_1(0) = 0 \quad y_1'(0) = 0$

Solving

$$y(t,\varepsilon) = \sin t - \tfrac{1}{2}\varepsilon t \sin t \quad + O(\varepsilon^2)$$

Though $y - y_0 - \varepsilon y_1 = O(\varepsilon^2)$ uniformly on $[0, T]$ the approximation is poor for several reasons.

Secularity

An old term from astronomy. Small changes in orbits over long times were referred to as secularities.

$$y(t,\varepsilon) = \sin t - \underbrace{\tfrac{1}{2}\varepsilon t \sin t}_{\substack{\text{secular and} \\ \text{changes order} \\ \text{if } t = O(1/\varepsilon)}} + O(\varepsilon^2)$$

Qualitative behavior

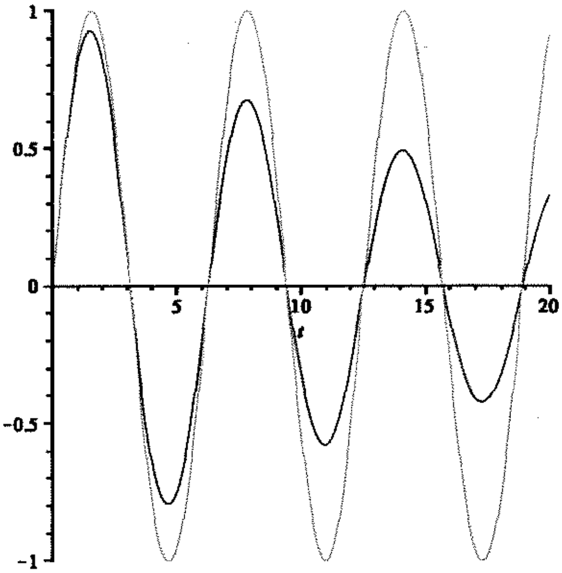
The exact solution decays whereas the two term regular expansion grows as t gets large, i.e. $t \sin t$.

Dissipation

Original problem is dissipative whereas the leading regular problem is conservative.

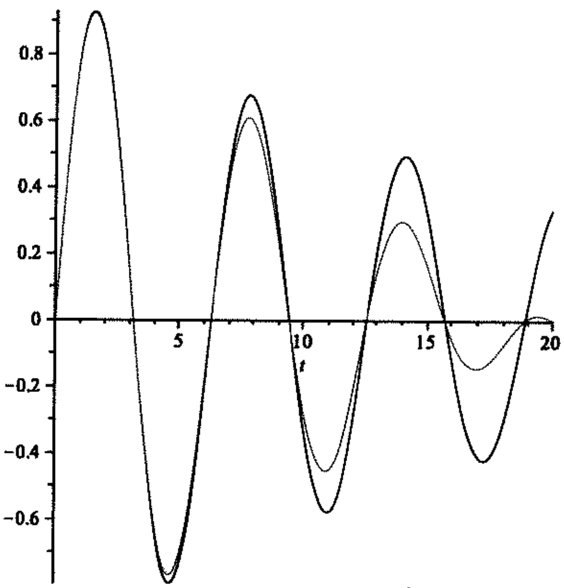
Amplitude Phase

Approximation $y_0 + \varepsilon y_1$ does a bad job capturing slowly varying phase.



$y_0(t)$ vs $y(t,\varepsilon)$

$\varepsilon = 0.1$



$y_0 + \varepsilon y_1$ vs $y(t,\varepsilon)$

better here

still not good here.

Dissipation (detail)

(1) $y'' + \varepsilon y' + y = 0 \qquad y(0) = 0 \quad y'(0) = 1$

Hamiltonian for unperturbed problem

$$H(y, y') = \tfrac{1}{2}(y')^2 + \tfrac{1}{2}y^2$$

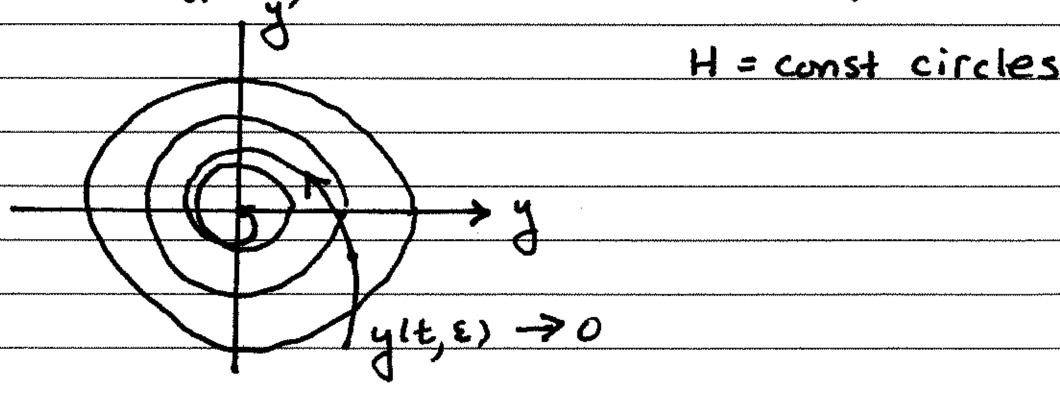
Multiply (1) by y' and integrate using I. Cond

$$H(y, y') = H(0, 1) - \varepsilon \int_0^t [y'(\tau)]^2 \, d\tau$$

Most importantly note that

$$\frac{dH}{dt} = -\varepsilon (y')^2 < 0$$

So that energy H is lost - hence dissipative.



But for $y_0 = \sin t$, $\dot{H} = 0 \Rightarrow$ conservative

Multiple Scales expansion

$$y(t,\varepsilon) = Y(t,\tau,\varepsilon) \equiv \frac{1}{\sqrt{1-\varepsilon^2/4}}\, e^{-\frac{1}{2}\tau} \sin\left(t\sqrt{1-\varepsilon^2/4}\right)$$

Expand in ε with both t and $\tau \equiv \varepsilon t$ fixed

$$Y(t,\tau,\varepsilon) = Y_0(t,\tau) + \varepsilon Y_1(t,\tau) + \cdots$$

Attached Maple code

$$Y_0(t,\tau) = Y(t,\tau,0)$$

and

$$Y_k(t,\tau) = \frac{1}{k!}\frac{\partial^k Y}{\partial \varepsilon^k}(t,\tau,0)$$

Here $Y_k = 0$ if k odd (code) and

$$Y_0 = e^{-\frac{1}{2}\tau}\sin t$$

$$Y_2 = \frac{1}{8}e^{-\frac{1}{2}\tau}(\sin t - \overset{\text{secular}}{t\cos t})$$

so that

$$Y = Y_0 + \varepsilon^2 Y_2 + O(\varepsilon^4) \qquad t = O(1/\varepsilon)$$

Exact solution of IVP:

```
> y:=t->1/sqrt(1-epsilon^2/4)*exp(-epsilon*t/2)*sin(t*sqrt(1-
  epsilon^2/4));
```

$$y := t \to \frac{e^{-\frac{1}{2}\varepsilon t}\sin\left(t\sqrt{1-\frac{1}{4}\varepsilon^2}\right)}{\sqrt{1-\frac{1}{4}\varepsilon^2}} \tag{1}$$

Re-express the exact solution as an exact TWO-TIME representation: $\tau = \varepsilon t$

```
> Y:=(t,tau)->1/sqrt(1-epsilon^2/4)*exp(-tau/2)*sin(t*sqrt(1-
  epsilon^2/4)):Y(t,tau);
```

$$\frac{2\,e^{-\frac{1}{2}\tau}\sin\left(\frac{1}{2}t\sqrt{4-\varepsilon^2}\right)}{\sqrt{4-\varepsilon^2}} \tag{2}$$

Fixing t and tau we exapnd in ε**:**

$Y(t,\tau,\varepsilon) = Y_0(t,\tau) + \varepsilon\,Y_1(t,\tau) + \varepsilon^2\,Y_2(t,\tau) + \varepsilon^3\,Y_3(t,\tau) + O(\varepsilon^4)$

then find $Y_k(t,\tau)$.

```
> YM[0]:=simplify(subs(epsilon=0,Y(t,tau))):
> for k from 1 by 1 to 3 do YM[k]:=simplify(subs(epsilon=0,diff(Y
  (t,tau),epsilon$k)))/k! od:
```

```
> for i from 0 to 3 do YM[i] od;
```

$$e^{-\frac{1}{2}\tau}\sin(t)$$

$$0$$

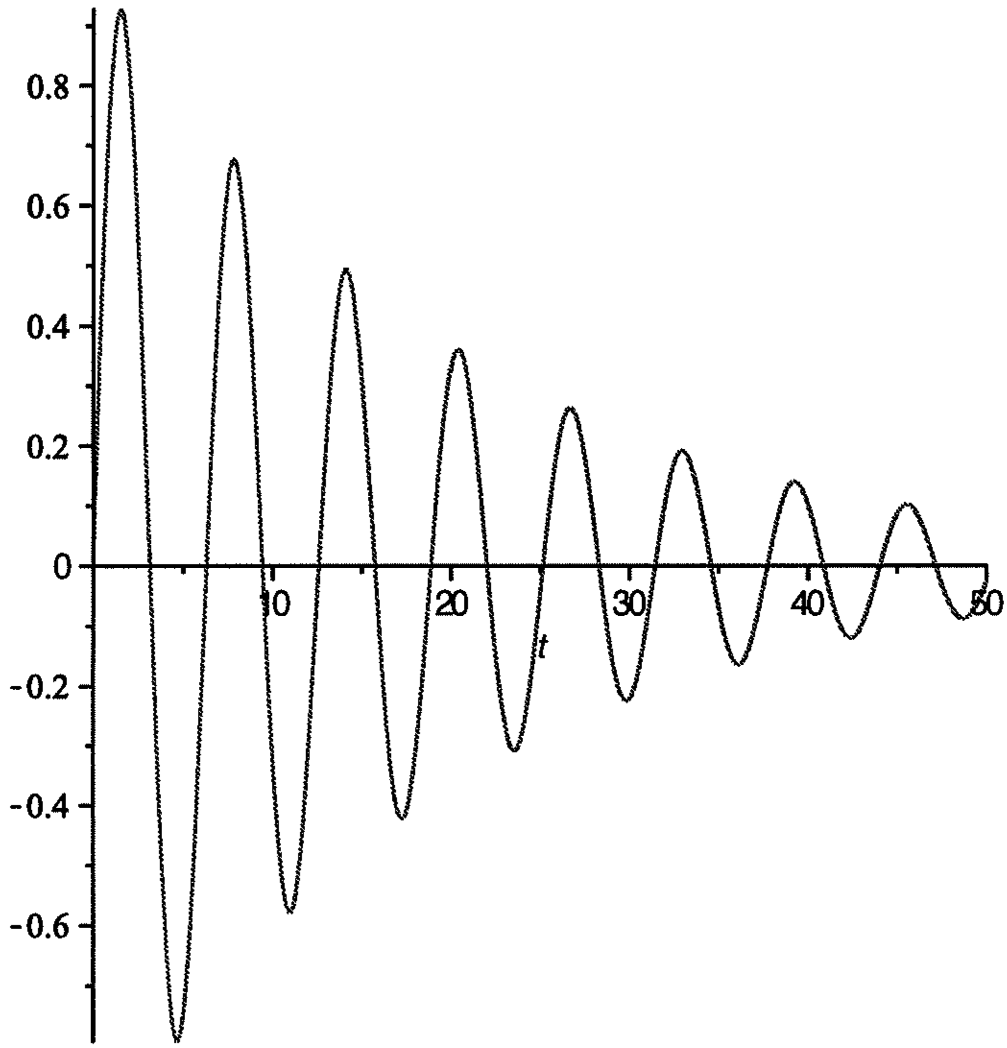
$$\frac{1}{8}e^{-\frac{1}{2}\tau}(\sin(t)-\cos(t)\,t)$$

$$0 \tag{3}$$

```
> P:=ep->plot([subs(epsilon=ep,y(t)),subs(epsilon=ep,subs(tau=
  epsilon*t,YM[0]))],t=0..50,color=[blue,red]);
```

$$P := ep \to plot\left(\left[subs(\varepsilon = ep, y(t)),\ subs\left(\varepsilon = ep,\ subs\left(\tau = \varepsilon t, YM_0\right)\right)\right], t = 0..50, color = [blue, red]\right) \tag{4}$$

```
> P(0.1);
```


0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
10
20
30
40
50
t

Multiple scales and strained coordinates

A strained coordinate $\tau(t,\varepsilon)$ such as

$\tau = t$	no straining
$\tau = \varepsilon t$	scaling
$\tau = \varepsilon^2 t$	scaling
$\tau = (1+\omega_1\varepsilon+\varepsilon^2\omega_2+\cdots)\,t$	Lindstedt straining
$\dfrac{d\tau}{dt} = \omega(\tilde{t}) \quad \tilde{t}=\varepsilon t$	Straining via differential eqns.

In a multiple scales expansion one assumes

$$(1) \qquad y(t,\varepsilon) = Y(\tau_1,..\tau_K,\varepsilon)$$

and

$$(2) \qquad Y \sim Y_0(\tau_1,..\tau_K) + \nu_1(\varepsilon)\,Y_1(\tau_1,..\tau_K) + o(\nu_1)$$

Expansions in (2) must be <u>consistent</u>.
That is to say:

For all $\tau_K = O(1)$, the term $\nu_1 Y_1 = O(\nu_1)$ when re written in terms of t.

EXAMPLES OF INCONSISTENT TERMS

$$y(t,\varepsilon) = Y_0(\tau,\tilde{t}) + \underbrace{\varepsilon Y_1(\tau,\tilde{t})}_{} + O(\varepsilon^2)$$

terms in here that are $O(1)$ upon renaming are inconsistent.

EXAMPLE $\tau = t$, $\tilde{t} = \varepsilon t$

$$y = \sin\tau + \varepsilon\tau\sin\tau + O(\varepsilon^2)$$

$$y = (1+\tilde{t})\sin\tau + O(\varepsilon^2)$$

EXAMPLE $\tau = t$, $\tilde{t} = \varepsilon t$

$$y = \varepsilon\tau^2\sin(\tilde{t}) + O(\varepsilon^2)$$

$$y = \tau\tilde{t}\sin(\tilde{t}) + O(\varepsilon^2)$$

EXAMPLE of a consistent term

$$T = \varepsilon \underbrace{\frac{\tau}{\tau^2+1}}_{} \sin\tau$$

Bounded (uniformly) for $\tau \in \mathbb{R}$.

Weakly nonlinear problems

$$(1) \qquad y'' + y = \varepsilon f(y, y')$$

$$(2) \qquad y(0) = a \qquad y'(0) = b$$

Since (1) is linear if $\varepsilon = 0$ but nonlinear otherwise, the problem is said to be weakly nonlinear

$$\tau_1 = t \qquad \tau_2 = \varepsilon t$$

are fast and slow strained times. Let

$$y(t,\varepsilon) = Y(\tau_1, \tau_2, \varepsilon) = Y_0(\tau_1, \tau_2) + \varepsilon Y_1(\tau_1, \tau_2) + O(\varepsilon^2)$$

To use this multiple time scale in (1)-(2) we must convert $(\;)'$ into partials involving τ_1, τ_2

$$(3) \qquad \frac{d}{dt} Y = \frac{\partial Y}{\partial \tau_1} + \varepsilon \frac{\partial Y}{\partial \tau_2}$$

$$(4) \qquad \frac{d^2}{dt^2} Y = \frac{\partial^2 Y}{\partial \tau_1^2} + 2\varepsilon \frac{\partial^2 Y}{\partial \tau_1 \partial \tau_2} + \varepsilon^2 \frac{\partial^2 Y}{\partial \tau_2^2}$$

These must also to derive I. Cond. for $Y_K(\tau_1, \tau_2)$

Generally

(5) $y(0,\varepsilon) = a = Y_0(0,0) + \varepsilon Y_1(0,0) + O(\varepsilon^2)$

(6) $y'(0,\varepsilon) = b = \frac{\partial Y_0}{\partial \tau_1}(0,0) + \varepsilon\left(\frac{\partial Y_1}{\partial \tau_1}(0,0) + \frac{\partial Y_0}{\partial \tau_2}(0,0)\right) + O(\varepsilon^2)$

Note the I.C. for Y_1 depend on Y_0!
Also

(7) $$f(y,y') = f\left(Y_0, \frac{\partial Y_0}{\partial \tau_1}\right) + O(\varepsilon)$$

Next we define the operator

(8) $$L(Y) \equiv \frac{\partial^2 Y}{\partial \tau_1^2} + Y$$

Collectively (1)-(8) yield an $O(1)$ and $O(\varepsilon)$ problem.

$$\boxed{\begin{aligned} L(Y_0) &= 0 \\ L(Y_1) &= f\left(Y_0, \frac{\partial Y_0}{\partial \tau_1}\right) - 2\frac{\partial^2 Y_0}{\partial \tau_1 \partial \tau_2} \end{aligned}}$$

with I. Conditions

$$Y_0(0,0) = a \qquad \frac{\partial Y_0}{\partial \tau_1}(0,0) = b$$

$$Y_1(0,0) = 0 \qquad \frac{\partial Y_1}{\partial \tau_1}(0,0) = -\frac{\partial Y_0}{\partial \tau_2}(0,0)$$

The general amplitude-phase solution of $O(1)$ problem

$$Y_0(\tau_1,\tau_2) = A(\tau_2)\cos(\tau_1 - \phi(\tau_2))$$

Initial conditions imply

$$A(0)\cos\phi(0) = a$$

$$A(0)\sin\phi(0) = b$$

From which we deduce

$$A(0) = \sqrt{a^2+b^2} \qquad \phi(0) = \arctan\left(\frac{b}{a}\right)$$

At this stage we still don't know $A(\tau_2)$, $\phi(\tau_2)$.

We require the expansion

$$Y \sim Y_0(\tau_1,\tau_2) + \varepsilon Y_1(\tau_1,\tau_2) + O(\varepsilon^2)$$

to be consistent for $0 \leq \tau_2 = O(1)$ or for $0 \leq \varepsilon t \leq T$ with T fixed.

One way to assure this is to require Y_K be periodic in τ_1 and hence uniformly bounded.

Defining

$$F_1(\tau_1,\tau_2) \equiv f\left(Y_0, \frac{\partial Y_0}{\partial \tau_1}\right) - 2\frac{\partial^2 Y_0}{\partial \tau_1 \partial \tau_2}$$

Fredholm Alternative - periodic case

$$L[u] \equiv u'' + a(t)u' + b(t)u$$

$$L^*[v] \equiv v'' - (a(t)v)' + b(t)v$$

The problem $L[u] = f(t)$ has T-periodic solutions only if

$$\langle v, f \rangle \equiv \int_0^T v(t) f(t)\, dt = 0 \qquad \forall v \in N(L^*)$$

Necessary conditions for $A(\tau_2)$ and $\phi(\tau_2)$

Given the F.A.T. above, Y_1 is 2π-periodic in τ_1 only if

$$\text{(a)} \qquad \int_0^{2\pi} F(\tau_1, \tau_2) \cos\tau_1 \, d\tau_1 = 0$$

$$\text{(b)} \qquad \int_0^{2\pi} F(\tau_1, \tau_2) \sin\tau_1 \, d\tau_1 = 0$$

Given how F is defined, these represent a coupled set of (nonlinear) differential equations for A, ϕ. Solving these with known I.C. completes to leading solution Y_0.

EXAMPLE $\boxed{y'' + y = -\varepsilon y^3 \qquad y(0) = 0 \qquad y'(0) = v > 0}$

Let $\tau_1 = t$, $\tau_2 = \varepsilon t$ and

$$y(t,\varepsilon) = Y_0(\tau_1,\tau_2) + \varepsilon Y_1(\tau_1,\tau_2) + O(\varepsilon^2)$$

The $O(1)$ and $O(\varepsilon)$ problems are

$$O(1) \quad \frac{\partial^2 Y_0}{\partial \tau_1^2} + Y_0 = 0 \qquad Y_0(0,0) = 0 \quad \frac{\partial Y_0}{\partial \tau_1}(0,0) = v$$

$$O(\varepsilon) \quad \frac{\partial^2 Y_1}{\partial \tau_1^2} + Y_1 = -Y_0^3 - 2\frac{\partial^2 Y_0}{\partial \tau_1 \partial \tau_2}, \quad Y_1(0,0) = 0 \quad \frac{\partial Y_1}{\partial \tau_1}(0,0) = -\frac{\partial Y_0}{\partial \tau_2}$$

From the $O(1)$ problem we deduce

(3) $\quad Y_0(\tau_1,\tau_2) = A(\tau_2)\cos(\tau_1 - \phi(\tau_2)) = A\cos\psi$

(4) $\quad A(0) = v$

(5) $\quad \phi(0) = \frac{\pi}{2}$

Compute RHS of $O(\varepsilon)$ equation

$$F_1 = -Y_0^3 - 2\frac{\partial^2 Y_0}{\partial \tau_1 \partial \tau_2}$$

$$F_1 = -A^3\cos^3\psi + 2A'\sin\psi - 2A\phi'\cos\psi$$

where $\psi = \tau_1 - \phi(\tau_2)$.

After some calculations

$$\langle F_1, \sin\psi \rangle = \int_0^{2\pi} F_1 \sin\psi \, d\psi = 2\pi A'$$

$$\langle F_1, \cos\psi \rangle = \int_0^{2\pi} F_1 \cos\psi \, d\psi = -\tfrac{3}{4}\pi A^3 - 2\pi A \phi'$$

For expansion to be consistent we require

(6) $\quad A' = 0 \qquad A(0) = v$

(7) $\quad \tfrac{3}{4}A^3 + 2A\phi' = 0 \qquad \phi(0) = \tfrac{\pi}{2}$

whose solution is

$$A(\tau_2) = v$$

$$\phi(\tau_2) = \frac{\pi}{2} - \frac{3}{8}v^2\tau_2$$

Thus, for $t = O(\frac{1}{\varepsilon})$

$$y(t,\varepsilon) = Y_0(\tau_1,\tau_2) = v\cos\left(\tau_1 - \frac{\pi}{2} + \frac{3}{8}v^2\tau_2\right) + O(\varepsilon)$$

Stated otherwise

$$y(t,\varepsilon) = v\sin\left(\left(1 + \frac{3}{8}v^2\varepsilon\right)t\right) + O(\varepsilon)$$

Change only in period, not amplitude.

Solvability conditions, consistency

$$y'' + y = \varepsilon f(y, y')$$

$$y(0) = a \qquad y'(0) = b$$

Multiple scales expansion

(1) $$y(t,\varepsilon) = Y(\tau_1, \tau_2, \varepsilon) = Y_0(\tau_1, \tau_2) + \varepsilon Y_1(\tau_1, \tau_2) + \cdots$$

where $\tau_1 = t$ and $\tau_2 = \varepsilon t$.

$O(1)$ $\quad L(Y_0) = 0 \qquad Y_0(0,0) = a \quad \frac{\partial Y_0}{\partial \tau_1}(0,0) = b$

$O(\varepsilon)$ $\quad L(Y_1) = F_1$

where

$$F_1 = f(Y_0, Y_0') - 2\frac{\partial^2 Y_0}{\partial \tau_1 \partial \tau_2}$$

Leading solution

$$Y_0 = A(\tau_2)\cos(\tau_1 - \phi(\tau_2))$$

can alternately be written

(2) $$Y_0 = A_0(\tau_2)\sin\tau_1 + B_0(\tau_2)\cos\tau_1$$

Consistency of expansion assured if

i) Y_1 bounded in τ_1

ii) Y_1 periodic in τ_1

Some authors assure consistency by

a) removing secular terms
b) applying Fred. Alternative

EXAMPLE $f(y) = y$

Using (2) in $O(\varepsilon)$ problem

$$L(Y_1) = \underbrace{(A_0 + 2B_0')}_{S_1} \sin \tau_1 + \underbrace{(B_0 - 2A_0')}_{S_2} \cos \tau_1$$

each indicated term generates secularities

$$Y_1 = A_1 \sin \tau_1 + B_1 \cos \tau_1 + \underbrace{2S_1 \tau_1 \sin \tau_1}_{\text{secular}} + \underbrace{2S_2 \tau_2 \cos \tau_1}_{\text{secular}}$$

Thus one must choose

$$\boxed{A_0 + 2B_0' = 0 \qquad B_0 - 2A_0' = 0}$$

Alternately these are the $k=1$ Fourier coefficients of F_1

$$F_1 = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\tau_1 + b_k \sin k\tau_1)$$

To eliminate those modes is

$$\int_0^{2\pi} F_1 \cos \tau_1 \, d\tau_1 = \int_0^{2\pi} F_1 \sin \tau_1 \, d\tau_1 = 0$$

which is the Fredholm alternative!

EXAMPLE (with dissipation)

$$\boxed{\begin{array}{cc} y'' + y = -\varepsilon (y')^3 & \\ y(0) = 1 & y'(0) = 0 \end{array}}$$

Let $\tau = t$, $\hat{t} = \varepsilon t$ and

$$(1) \qquad y(t, \varepsilon) = Y(\tau, \hat{t}, \varepsilon) = Y_0(\tau, \tilde{t}) + \varepsilon Y_1(\tau, \tilde{t}) + \cdots$$

Define the operator

$$L(Y) = \frac{\partial^2 Y}{\partial \tau^2} + Y$$

Derivative conversions

$$\frac{d}{dt} Y = \frac{\partial Y}{\partial \tau} + \varepsilon \frac{\partial Y}{\partial \hat{t}}$$

$$\frac{d^2}{dt^2} Y = \frac{\partial^2 Y}{\partial \tau^2} + 2\varepsilon \frac{\partial^2 Y}{\partial \tau \partial \hat{t}} + \varepsilon^2 \frac{\partial^2 Y}{\partial \tilde{t}^2}$$

Use these and (1) we arrive at

$O(1)$ $\qquad L(Y_0) = 0$

$O(\varepsilon)$ $\qquad L(Y_1) = F_1$

$O(\varepsilon^2)$ $\qquad L(Y_2) = F_2$

where,

$$F_1 = -2\frac{\partial^2 Y_0}{\partial\tau\partial\tilde{t}} - \left(\frac{\partial Y_0}{\partial\tau}\right)^3$$

$$F_2 = -2\frac{\partial^2 Y_1}{\partial\tau\partial\tilde{t}} - \frac{\partial^2 Y_0}{\partial\tilde{t}^2} - 3\left(\frac{\partial Y_0}{\partial\tau}\right)^2\frac{\partial Y_1}{\partial\tau} - 3\left(\frac{\partial Y_0}{\partial\tau}\right)^2\frac{\partial Y_0}{\partial\tilde{t}}$$

<u>O(1) Problem</u>

$$Y_0(t,\tilde{t}) = A_0(\tilde{t})\cos(\tau+\phi_0(\tilde{t}))$$

is amplitude-phase form. Initial conditions

$$Y_0(0,0) = A_0(0)\cos\phi_0(0) = 1$$

$$\frac{\partial Y_0}{\partial\tau}(0,0) = -A_0(0)\sin\phi_0(0) = 0$$

from which we conclude

$$A_0(0) = 1 \qquad \phi_0(0) = 0$$

At this stage we do not know $A_0(\tilde{t})$ and $\phi_0(\tilde{t})$

$$\boxed{Y_0(\tau,\tilde{t}) = A_0(\tilde{t})\cos\psi}$$

where $\psi = \tau + \phi_0(\tilde{t})$

$O(\varepsilon)$ problem determines A_0, ϕ_0

$$L(Y_1) = 2A_0' \sin\psi + 2A_0\phi_0' \cos\psi + A_0^3 \sin^3\psi = F_1$$

To eliminate inconsistent terms in Y_1 we require it to be periodic in τ. Fredholm alternative

$$\langle F_1, \sin\psi\rangle = 0 \qquad A_0' + \frac{3}{8}A_0^3 = 0$$

$$\langle F_1, \cos\psi\rangle = 0 \qquad 2A_0\phi_0' = 0$$

Solving the ODE system for (A_0, ϕ_0) with derived initial conditions

$$A_0(\tilde{t}) = \frac{2}{\sqrt{3\tilde{t}+4}} \qquad \phi_0(\tilde{t}) = 0$$

and

$$(2) \qquad Y_0(\tau, \tilde{t}) = \frac{2}{\sqrt{3\tilde{t}+4}} \cos\tau$$

Simplify $O(\varepsilon)$ problem

Using $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ in F_1 and (2) we get

$$F_1 = -\frac{2\sin(3\psi)}{(3\tilde{t}+4)^{3/2}}$$

Note absence of $\sin\psi$, $\cos\psi$ Fourier components.

Solving for $\Upsilon_1(\tau, \tilde{t})$:

$$(3) \qquad \Upsilon_1 = \underbrace{A_1(\tilde{t})\cos(\tau + \phi_1(\tilde{t}))}_{\text{amplitude phase part}} + \frac{\sin 3\tau}{4(3\tilde{t}+4)^{3/2}}$$

Notice Υ_1 is bounded and periodic in τ. More over $\Upsilon_1 = O(1)$ for $\tilde{t} = O(1)$, i.e. long times.

<u>$O(\varepsilon^2)$ outline</u> $\qquad \psi \equiv \tau + \phi_1$

$$(4) \qquad L(\Upsilon_2) = F_2(\Upsilon_0, \Upsilon_1)$$

where we suppress arguements involving the derivatives of Υ_K.

Knowing Υ_0 completely and using (3) in (4) the resulting consistency conditions

$$\langle F_2, \sin\psi \rangle = 0$$

$$\langle F_2, \cos\psi \rangle = 0$$

yield coupled ODE's for A_1 and ϕ_1. Using ininitial cond. to omplete Υ_1,

$$\Upsilon = \Upsilon_0(\tau, \tilde{t}) + \varepsilon\Upsilon_1(\tau, \tilde{t}) + O(\varepsilon^2)$$

for $\tilde{t} = O(1)$ or $t = O(1/\varepsilon)$.

Multiple Scales Linear Straining

(1) $\quad y'' + y = \varepsilon f(y, y') \qquad y(0) = A \quad y'(0) = B$

Seek expansion

$$y \sim \sum_{n=0}^{\infty} Y_n(\tau, \tilde{t})$$

$$\tau = \sum_{n=0}^{\infty} \omega_n \varepsilon^n \qquad \omega_0 = 1 \qquad \text{(linear strain)}$$

$$\tilde{t} = \varepsilon t \qquad \text{slow time}$$

The slight generalization in τ embodies Linsteadt's method for finding periodic orbits. Then $\tilde{t}$ is absent.

$$\frac{dy}{dt} = \frac{\partial Y_0}{\partial \tau} + \varepsilon \left(\frac{\partial Y_1}{\partial \tau} + \frac{\partial Y_0}{\partial \tilde{t}} + \omega_1 \frac{\partial Y_1}{\partial \tau} \right) + O(\varepsilon^2)$$

Solution of $O(1)$ problem

$$L(Y_0) = 0 \qquad Y_0(0,0) = A, \quad \frac{\partial Y_0}{\partial \tau}(0,0) = B$$

is

$$Y_0 = A_0(\tilde{t}) \cos \psi \qquad \psi \equiv \tau - \phi_0(\tilde{t})$$

where $A_0(0)$ and $\phi_0(0)$ are known.

Inconsistent terms to $O(\varepsilon)$

$$y'' = Y_{0\tau\tau} + \varepsilon\left(Y_{1\tau\tau} + 2\omega_1 Y_{0\tau\tau} + 2Y_{0\tau\tilde{t}}\right) + O(\varepsilon^2)$$

The $O(\varepsilon)$ equation is

$$L(Y_1) = F_1 = -2\omega_1 Y_{0\tau\tau} - 2Y_{0\tau\tilde{t}} + f^{(0)}$$

Given soln $Y_0 = A_0 \cos\psi$ one finds

$$F_1(\psi, \tilde{t}) = 2A_0' \sin\psi - 2A_0(\phi_0' - \omega_1)\cos\psi + f^{(0)}$$

Fourier expand $f^{(0)}$ to remove first harmonics.

$$P_1(\underset{\uparrow}{A_0}) = \langle f^{(0)}, \sin\psi\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\underset{\uparrow}{A_0}\cos\psi, -\underset{\uparrow}{A_0}\sin\psi)\sin\psi \, d\psi$$

$$Q_1(A_0) = \langle f^{(0)}, \cos\psi\rangle$$

Hence

$$f^{(0)} = \underset{\substack{\uparrow \\ \text{doesn't contribute} \\ \text{to inconsistent} \\ \text{terms.}}}{\hat{f}^{(0)}} + 2P_1 \sin\psi + 2Q_1 \cos\psi$$

Then F_1 may be written

$$F_1 = 2(\underbrace{A_0' + P_1}_{0})\sin\psi + 2(\underbrace{-A_0\phi_0' + A_0\omega_1 + Q_1}_{0})\cos\psi + \hat{f}^{(0)}$$

Results in ODE's

$$\frac{dA_0}{d\tilde{t}} = -P_1(A_0)$$

$$A_0\left(\frac{d\phi_0}{d\tilde{t}} - \omega_1\right) = Q_1(A_0)$$

Solution for amplitude given implicitly by

$$\tilde{t} = -\int_{A_0(0)}^{A_0(\tilde{t})} \frac{ds}{P_1(s)}$$

Inverted to find $A_0(\tilde{t})$ one then gets

$$\phi(\tilde{t}) = \phi_0(0) + \omega_1\tilde{t} + \int_0^{\tilde{t}} \frac{Q_1(A_0(s))}{A_0(s)}\,ds$$

<u>Remarks (see Kevorkian 95)</u>

i) $\omega_1 = 0$ wlog

ii) General expressions to $O(\varepsilon^2)$ possible but $\omega_2 \neq 0$

Fixed Period Straining

$\tilde{t} = \varepsilon t$ slow time

$$\boxed{\begin{array}{l} y'' + \omega^2(\tilde{t})\, y = \varepsilon f(y, y', \tilde{t}) \\ y(0) = A \qquad y'(0) = B \end{array}}$$

Multiple scales

$$y(t, \varepsilon) = Y(\tau, \tilde{t}, \varepsilon) = Y_0(\tau, \tilde{t}) + \varepsilon Y_1(\tau, \tilde{t}) + \cdots$$

where

$$\frac{d\tau}{dt} = \omega(\tilde{t}) \qquad \tau = \int_0^t \omega(\varepsilon\eta)\, d\eta$$

is a strained fast time.

Derivative Conversions, etc

$$y' = \omega \frac{\partial Y}{\partial \tau} + \varepsilon \frac{\partial Y}{\partial \tilde{t}}$$

$$y'' = \omega^2 \frac{\partial^2 Y}{\partial \tau^2} + 2\varepsilon\omega \frac{\partial^2 Y}{\partial \tau \partial \tilde{t}} + \varepsilon \omega' \frac{\partial Y}{\partial \tau} + \varepsilon^2 \frac{\partial^2 Y}{\partial \tilde{t}^2}$$

$$f = f\left(Y_0, \omega \frac{\partial Y_0}{\partial \tau}, \tilde{t}\right) + O(\varepsilon) \equiv f^{(0)} + O(\varepsilon)$$

Initial Conditions

$$Y_0(0,0) = A \qquad \frac{\partial Y_0}{\partial \tau}(0,0) = \frac{B}{\omega(0)}$$

$$Y_1(0,0) = 0 \qquad \frac{\partial Y_1}{\partial \tau}(0,0) = \frac{1}{\omega(0)} \frac{\partial Y_0}{\partial \tilde{t}}(0,0)$$

$O(1)$ and $O(\varepsilon)$ problems

$$L(\Psi_0) = \frac{\partial^2 \Psi_0}{\partial \tau^2} + \Psi_0 = 0$$

has same form

$$\Psi_0 = A_0(\tilde{t}) \cos \psi \qquad \psi = \tau - \phi(\tilde{t})$$

$O(\varepsilon)$ problem more complex

$$L(\Psi_1) = -\frac{2}{\omega} \frac{\partial^2 \Psi_0}{\partial \tau \partial \tilde{t}} - \frac{\omega'}{\omega^2} \frac{\partial \Psi_0}{\partial \tau} + \frac{1}{\omega^2} f^{(0)} = F_1$$

As before $\langle F_1, \sin\psi \rangle = \langle F_1, \cos\psi \rangle$ assures consistency.

PENDULUM EQN: $\omega^2 \propto \frac{1}{L}$ where L = length.

EXAMPLE $y'' + \omega^2(\tilde{t})\, y = 0$ $\quad y(0) = A \quad y'(0) = B$

O(1) PROBLEM $\quad L[\Upsilon_0] = 0 \quad \Upsilon_0(0,0) = A \quad \frac{\partial \Upsilon_0}{\partial \tau}(0,0) = \frac{B}{\omega(0)}$

$$\Upsilon_0(\tau, \tilde{t}) = A_0(\tilde{t}) \cos \psi \qquad \psi = \tau - \phi_0(\tilde{t})$$

$$A_0(0) = \sqrt{A^2 + \frac{B^2}{\omega(0)^2}}$$

$$\phi_0(0) = \arctan\left(\frac{B}{A\omega(0)}\right)$$

O(ε) PROBLEM Straining alone induces inconsistencies

$$L[\Upsilon_1] = \frac{1}{\omega^2}(2A_0'\omega + \omega' A_0)\sin\psi - \frac{2A_0}{\omega}\phi_0' \cos\psi$$

Removal of secular terms (inconsist)

$$\phi_0' = 0 \qquad \phi_0(\tilde{t}) = \phi_0(0)$$

$$2A_0'\omega + \omega' A_0 = 0 \qquad A_0(\tilde{t}) = A_0(0)\sqrt{\frac{\omega(0)}{\omega(\tilde{t})}}$$

Thus for $\omega(\tilde{t}) = 1 + \tilde{t}$ linear ↑ in frequency

$$\tau = \int_0^t \omega(\varepsilon\eta)\, d\eta = t + \tfrac{1}{2}\varepsilon t^2$$

$$\Upsilon_0(\tau, \tilde{t}) = A_0(\tilde{t}) \cos(\tau - \phi_0(0))$$

$$\Upsilon_0(\tau, \tilde{t}) = A_0(0) \frac{1}{\sqrt{1 + \varepsilon t}} \cos(t + \tfrac{1}{2}\varepsilon t^2 - \phi_0(0))$$

EXAMPLE

$$\boxed{\begin{array}{lll} y'' + \omega^2(\varepsilon t) y = -\varepsilon y' & & \omega(s) = 1+s \\ y(0) = a & y'(0) = b & \end{array}}$$

Solving with maple code

$$A(\tau_2) = \frac{A(0) e^{\frac{1}{2}\tau_2}}{\sqrt{\tau_2 + 1}}$$

$$\phi(\tau_2) = \phi(0)$$

Giving the solution

$$y(t,\varepsilon) \sim \frac{A(0) e^{\frac{1}{2}\varepsilon t}}{\sqrt{1+\varepsilon t}} \cos\left(t + \frac{\varepsilon t^2}{2} - \phi(0) \right)$$

The strained time

$$\tau_1 = \int_0^t (1+\varepsilon s)\, ds = t + \frac{\varepsilon t^2}{2}$$

EXAMPLE

$$\boxed{\begin{array}{ll} y'' + \omega^2(\varepsilon t)\, y = -\varepsilon (y')^3 & \\ y(0) = a & y'(0) = b \end{array}}$$

$\omega > 0$

O(1) Problem

$$Y_0(\tau_1, \tau_2) = A(\tau_2)\cos\psi \qquad \psi = \tau_1 - \phi(\tau_2)$$

$$A(0) = \pm\left(a^2 + \frac{b^2}{\omega(0)^2}\right)^{1/2} \qquad Y_0(0,0) = a$$

$$\phi_0(0) = \arctan\left(\frac{b}{a\,\omega(0)}\right) \qquad \frac{\partial Y_0}{\partial \tau_1}(0,0) = \frac{b}{\omega(0)}$$

O(ε) Problem

$$L[Y_1] = F_1[Y_0] = -\frac{2}{\omega}\frac{\partial^2 Y_0}{\partial\tau_1\partial\tau_2} - \frac{\omega'}{\omega^2}\frac{\partial Y_0}{\partial \tau_1} - \frac{1}{\omega^2}\left(\omega\frac{\partial Y_0}{\partial\tau_1}\right)^3$$

Fourier expand F_1 in $\psi = \tau_1 - \phi(\tau_2)$ slowly varying phase

$$\Phi_1 = \int_0^{2\pi} F_1 \sin\psi\, d\psi = \frac{\pi}{4\omega^2}\left\{ 8A'\omega + 4\omega' A - 3\omega^3 A^3 \right\} = 0$$

$$\Phi_2 = \int_0^{2\pi} F_1 \cos\psi\, d\psi = -\frac{2\pi}{\omega} A\phi' = 0$$

Thus $\phi(\tau_2) = \phi(0)$. Integrating Φ_1 equation gives

$$A(\tau_2) = \pm\frac{\omega(\tau_2)^{1/2}}{\sqrt{\dfrac{1}{\omega(0)A(0)^2} - \dfrac{3}{4}\displaystyle\int_0^{\tau_2}\omega(s)\,ds}}$$

completing soln for Y_0.

Addendum   Solution to $\Phi_1 = 0$ differential equation

$$2A'\omega + \omega' A - \frac{3}{4}\omega^3 A^3 = 0$$

$$(\omega A^2)' - \frac{3}{4}\omega^3 A^4 = 0$$

$$u' - \frac{3}{4}\omega u^2 = 0 \qquad u \equiv \omega A^2$$

$$\frac{du}{d\tau_2} = \frac{3\omega u^2}{4}$$

$$\int_0^{\tau_2} \frac{du}{u^2} = \int_0^{\tau_2} \frac{3\omega}{4}\, d\tau_2$$

$$\frac{1}{\omega A^2} = \left(\frac{1}{\omega_0 A_0^2} - \int_0^{\tau_2} \frac{3\omega(s)}{4}\, ds\right) \qquad \omega_0 \equiv \omega(0),\ A_0 \equiv A(0)$$

Solving yields

$$A(\tau_2) = \frac{\omega(\tau_2)^{1/2}}{\sqrt{\dfrac{1}{\omega_0^2 A_0^2} - \displaystyle\int_0^{\tau_2} \frac{3\omega(s)}{4}\, ds}}$$

Multiple Scales BVP

(1) $\quad \varepsilon y'' + 2y' + 2y = 0 \qquad y(0) = 0 \qquad y(1) = 1$

Expect BLayer of $O(\varepsilon)$ thickness

$$x_1 = \frac{x}{\varepsilon} \qquad x_2 = x$$

are two space scales.

$$y = Y(x_1, x_2, \varepsilon) \sim Y_0(x_1, x_2) + \varepsilon Y_1(x_1, x_2) + \cdots$$

Derivative conversion

$$y' = \frac{1}{\varepsilon} Y_{x_1} + Y_{x_2} = \frac{1}{\varepsilon} Y_{0x_1} + (Y_{1x_1} + Y_{0x_2}) + O(\varepsilon)$$

$$\varepsilon y'' = \frac{1}{\varepsilon} Y_{x_1x_1} + 2Y_{x_1x_2} + \varepsilon Y_{x_2x_2} = \frac{1}{\varepsilon} Y_{0x_1x_1} + (Y_{1x_1x_1} + 2Y_{0x_1x_2}) + O(\varepsilon)$$

Leading and $O(\varepsilon)$ problem

$O(1)$ $\quad \dfrac{\partial^2 Y_0}{\partial x_1^2} + 2\dfrac{\partial Y_0}{\partial x_1} = 0$

$O(\varepsilon)$ $\quad \dfrac{\partial^2 Y_1}{\partial x_1^2} + 2\dfrac{\partial Y_1}{\partial x_1} = -2\dfrac{\partial^2 Y_0}{\partial x_1 \partial x_2} - 2\dfrac{\partial Y_0}{\partial x_2} - 2Y_0$

General soln to $O(1)$ problem

$$Y_0(x_1, x_2) = a_0(x_2) + b_0(x_2)e^{-2x_1}$$

General solution of $O(\varepsilon)$ problem

$$Y_1 = a_1(x_2) + b_1(x_2)e^{-2x_1} - (a_0' + a_0)x_1 \quad + (b - b_0')x_1 e^{-2x_1}$$

$$\underbrace{\frac{x}{\varepsilon} \qquad \frac{x}{\varepsilon}e^{-2x/\varepsilon}}$$

inconsistent terms change order on [0,1] when reexpressed in terms of x

Conclude

$$a_0' + a_0 = 0 \qquad a_0 = \bar{a}_0 e^{-x_2}$$

$$b_0' - b_0 = 0 \qquad b_0 = \bar{b}_0 e^{x_2}$$

and

$$Y_0(x_1, x_2) = \bar{a}_0 e^{-x_2} + \bar{b}_0 e^{x_2 - 2x_1}$$

Boundary conditions

$$Y_0(0,0) = 0 \qquad Y_0(\tfrac{1}{\varepsilon}, 1) \sim Y_0(\infty, 1) = 1$$

Solution

$$Y_0(x_1, x_2) = e^{1-x_2} - e^{1+x_2-2x_1}$$

$$y(x,\varepsilon) \sim e^{1-x} + e^{1+x-\frac{2x}{\varepsilon}}$$