Algebraic Problem: $f(x,\varepsilon)=0$, $x\in\mathbb{R}$

Seek a regular solution

$$(1) \qquad x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

Then

$$F(\varepsilon) \equiv f(x(\varepsilon),\varepsilon) = 0 \qquad \forall \varepsilon \in N_r(0)$$

If $f(x,\varepsilon)$ is smooth then the Taylor series of $F(\varepsilon)$ about $\varepsilon=0$ must vanish.

$$F(0) + F'(0)\,\varepsilon + \frac{1}{2!}F''(0)\,\varepsilon^2 + \cdots = 0$$

Necessary conditions are $F^{(n)}(0)=0$, $\forall n \geqslant 0$.

Repeated use of chain rules:

$$(2) \quad F'(\varepsilon) = f_x \frac{dx}{d\varepsilon} + f_\varepsilon$$

$$(3) \quad F''(\varepsilon) = f_{xx}\left(\frac{dx}{d\varepsilon}\right)^2 + 2 f_{x\varepsilon}\frac{dx}{d\varepsilon} + f_x \frac{d^2x}{d\varepsilon^2} + f_{\varepsilon\varepsilon}$$

Next, using (1)

$$\frac{dx}{d\varepsilon} = x_1 + 2\varepsilon x_2 + O(\varepsilon^2)$$

$$\frac{d^2x}{d\varepsilon^2} = 2x_2 + O(\varepsilon)$$

Use these in (2)-(3) and set $\varepsilon=0$

The conditions $F'(0) = F''(0) = 0$ imply

$$O(\varepsilon) \qquad f_x\, x_1 \quad + \quad f_\varepsilon \qquad = 0$$

$$O(\varepsilon^2) \qquad f_x\, x_2 \quad + \frac{1}{2} f_{xx}\, x_1^2 + x_1 f_{x\varepsilon} + f_{\varepsilon\varepsilon} \quad = 0$$

where all partials of f are evaluated at $(x_0, 0)$

Solve for x_1, then x_2, ...

Recall that the condition $f_x(x_0, 0) \neq 0$ in the I.V.T. is need to assure the existence (and smoothness) of an $X(\varepsilon)$.

Considering the $O(\varepsilon)$ eqn (or $O(\varepsilon)$ term of $F(\varepsilon)$)

$$x_1 = -\frac{f_\varepsilon(x_0, 0)}{f_x(x_0, 0)}$$

is defined only if $f_x(x_0, 0) \neq 0$!!

In practice such general theory is not often used to find x_K. Especially hard if we seek to find x_3 !!

EXAMPLE Polynomials

$$(1) \qquad x^3 + \varepsilon x - 8 = 0$$

Here $x_0 = 2$ so that $f(2,0) = 0$. Also, $f_x(2,0) = 12 \neq 0$ so IFThm implies $x(\varepsilon)$ has the expansion

$$(2) \qquad x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$$

Substitute (2) into (1) and collect powers of ε.

$O(1)$ $\qquad x_0^3 - 8 = 0$

$O(\varepsilon)$ $\qquad 3x_0^2 x_1 - x_0 = 0$

$O(\varepsilon^2)$ $\qquad 3x_0^2 x_2 + 3x_1^2 x_0 - x_1 = 0$

Solve successively for x_0, x_1 then x_2.

$$x_0 = 2 \qquad x_1 = \tfrac{1}{6} \qquad x_2 = 0$$

and conclude

$$x = 2 + \tfrac{1}{6}\varepsilon + O(\varepsilon^3)$$

EXAMPLE   Binomial Theorem

For any $p \in \mathbb{C}$ and $|z| < 1$

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \cdots$$

is the complex version of the binomial Theorem.

$$f(x,\varepsilon) = (1+x)^{1/3} - \cos x + \frac{\varepsilon}{1-x} = 0$$

Note that $f(0,0) = 0$ so the expansion has the form

$$x = \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$$

Since $x = O(\varepsilon)$ one can use the binomial theorem.

$$(1+x)^{1/3} = 1 + \tfrac{1}{3}(\varepsilon x_1 + \varepsilon^2 x_2 + \cdot) - \tfrac{1}{9}(x_1\varepsilon + \varepsilon^2 x_2 + \cdot\cdot)^2 + O(\varepsilon^3)$$

$$\cos x = 1 - \tfrac{1}{2}(\varepsilon x_1 + \varepsilon^2 x_2 + \cdot\cdot)^2 + O(\varepsilon^4)$$

$$\frac{\varepsilon}{1-x} = \varepsilon\Big(1 + (x_1\varepsilon + \varepsilon^2 x_2 + \cdot) + O(\varepsilon^2)\Big)$$

Collect like powers of ε to yield.

$O(\varepsilon)$	$\tfrac{1}{3}x_1 + 1 = 0$	$x_1 = -3$
$O(\varepsilon^2)$	$\tfrac{1}{3}x_2 + \tfrac{7}{18}x_1^2 + x_1 = 0$	$x_2 = -\tfrac{3}{2}$

Collectively

$$x = -3\varepsilon - \tfrac{3}{2}\varepsilon^2 + O(\varepsilon^3)$$

EXAMPLE Taylor Series

$$f(x,\varepsilon) \equiv \cos(\varepsilon e^x) + \log(1+\varepsilon\cos x) - \sqrt{2-x}$$

Note that $f(x,0) = 1 - \sqrt{2-x}$ which has the root $x=1$. Hence

$$x = x_0 + \varepsilon x_1 + \cdots = 1 + \varepsilon x_1 + O(\varepsilon^2)$$

General Taylor Series theory $\Rightarrow$

$$(1) \qquad x_1 = \frac{f_\varepsilon(x_0,0)}{f_x(x_0,0)}$$

Still cumbersome but relatively direct and arguably better than power series methods.

$$(2) \quad f_x(x,\varepsilon) = -\varepsilon e^x \sin(\varepsilon e^x) - \frac{\varepsilon\sin(x)}{1+\varepsilon\cos x} + \frac{1}{2\sqrt{2-x}}$$

$$(3) \quad f_\varepsilon(x,\varepsilon) = -e^x\sin(\varepsilon e^x) + \frac{\cos x}{1+\varepsilon\cos x}$$

Evaluate (2)-(3) at $(x_0,0)=(1,0)$ and use in (1)

$$x_1 = 2\cos(1)$$

Systems of equations $f(x, \varepsilon) = 0$

Here $\bar{x} = (x_1, \ldots x_n) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.

$$(1) \qquad F(\varepsilon) \equiv f(\bar{x}(\varepsilon), \varepsilon)$$

is also a vector valued function. Each of its elements must have T-series which vanish to all orders. That is to say if

$$F_i(\varepsilon) = F_i(0) + F_i'(0)\,\varepsilon + \cdots$$

then $F_i^{(n)}(0) = 0$, $\forall n = 0, 1, 2 \ldots$

We assume

$$\bar{x}(\varepsilon) = \bar{x}_0 + \varepsilon\,\bar{x}_1 + \varepsilon^2\,\bar{x}_2 + O(\varepsilon^3)$$

where the vectors $\bar{x}_k = (x_{1,k}, \ldots x_{n,k})$.

To examine how one might derive general expressions, consider the i^{th} component of f

$$F_i(\varepsilon) = f_i(\bar{x}_0 + \varepsilon\bar{x}_1 + \cdots, \varepsilon)$$

A long hand derivative of this in ε yields

$$F_i'(\varepsilon) = \frac{\partial f_i}{\partial x_1}(x_{1,1} + O(\varepsilon)) + \cdots \frac{\partial f_i}{\partial x_n}(x_{1,n} + O(\varepsilon)) + \frac{\partial f_i}{\partial \varepsilon}$$

Evaluate at $\varepsilon = 0$

$$(2) \qquad F_i'(0) = \frac{\partial f_i}{\partial x_1} x_{1,1} + \cdots \frac{\partial f_i}{\partial x_n} x_{1,n} + \frac{\partial f_i}{\partial \varepsilon} \qquad \forall i = 1, \ldots n$$

The expression in (2) can be written in vector form as

$$F'(0) = Df(\bar{x}_0, 0)\,\bar{x}_1 + \frac{\partial f}{\partial \varepsilon}(\bar{x}_0, 0)$$

where the Jacobian (at $x = \bar{x}_0, \varepsilon = 0$)

$$Df(\bar{x}_0, 0) = \left[\frac{\partial f_i}{\partial x_j}\right]$$

Thus a two term expansion of (1) is

$$F(\varepsilon) = f(\bar{x}_0, 0) + \varepsilon\left(Df(\bar{x}_0, 0)\,\bar{x}_1 + \frac{\partial f}{\partial \varepsilon}(\bar{x}_0, 0)\right) + O(\varepsilon^2)$$

For the $O(1)$ and $O(\varepsilon)$ terms to vanish

$$\boxed{\begin{array}{lll} O(1) & f(\bar{x}_0, 0) & = 0 \\ O(\varepsilon) & \bar{x}_1 & = -Df(\bar{x}_0, 0)^{-1}\,\dfrac{\partial f}{\partial \varepsilon}(\bar{x}_0, 0) \end{array}}$$

One could use these expressions to sequentially find $\bar{x}_0$ then $\bar{x}_1$ but in practice such systems are found by expand the systems using a variety of tools.

Higher order expressions

For simplicit $f = f(\bar{x}, \varepsilon)$ where $\bar{x} = (x, y) \in \mathbb{R}^2$.

$$x = x_0 + \varepsilon x_1 + \cdots$$
$$y = y_0 + \varepsilon y_1 + \cdots$$

Consider the i-th component of f where $i = 1, 2$.

$$F_i(\varepsilon) \equiv f_i(x_0 + \varepsilon x_1 + \cdots, y_0 + \varepsilon y_1 + \cdots; \varepsilon)$$

After lengthy calculations one can show

$$\frac{1}{2} F_i''(0) = \frac{1}{2} \bar{x}_0^T H_{f_i} \bar{x}_0 + \left(\frac{\partial^2 f_i}{\partial x \partial \varepsilon}\right) x_1 + \left(\frac{\partial^2 f_i}{\partial y \partial \varepsilon}\right) y_1 + \left(\frac{\partial f_i}{\partial x}\right) x_2 + \left(\frac{\partial f_i}{\partial y}\right) y_2 + \frac{1}{2} \frac{\partial^2 f_i}{\partial \varepsilon^2}$$

where the Hessian is that of f_i in (x, y).
Collectively

$$F(\varepsilon) = f(\bar{x}_0, \varepsilon) + \varepsilon \left(Df(\bar{x}_0, 0) \bar{x}_1 + \frac{\partial f}{\partial \varepsilon}(\bar{x}_0, 0) \right)$$
$$+ \varepsilon^2 \left(Df_\varepsilon(\bar{x}_0, 0) \bar{x}_1 + Df(\bar{x}_0, 0) \bar{x}_2 + \frac{1}{2} \bar{x}_0 H_f \bar{x}_0 \; \frac{\partial^2 f}{\partial \varepsilon^2} \right)$$

where it is understood that

$$H_f = \begin{pmatrix} H_{f_1}(\bar{x}_0, 0) \\ \\ H_{f_2}(\bar{x}_0, 0) \end{pmatrix}$$

Though doable this is not pretty.

EXAMPLE

$$(1) \qquad \boxed{\begin{aligned} xy - 27 - \varepsilon\sqrt{x+1} &= 0 \\ x^2 - y - \varepsilon x &= 0 \end{aligned}}$$

We use the expansions

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

$$y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$$

in the system (1) and expand in $\varepsilon \ll 1$

$$(x_0 + \varepsilon x_1 + \cdots)(y_0 + \varepsilon y_1 + \cdots) - 27 - \varepsilon\sqrt{1 + x_0 + \varepsilon x_1 + \cdots} = 0$$

$$(x_0 + \varepsilon x_1 + \cdots)^2 - (y_0 + \varepsilon y_1 + \cdots) - \varepsilon(x_0 + \varepsilon x_1 + \cdots) = 0$$

Collecting in powers of ε yields $O(1)$ and $O(\varepsilon)$ problems

$$O(1) \qquad \boxed{\begin{aligned} x_0 y_0 - 27 &= 0 \\ x_0^2 - y_0 &= 0 \end{aligned}} \qquad \text{"leading"}$$

whose soln by inspection is $x_0 = 3$, $y_0 = 9$

$$O(\varepsilon) \qquad \boxed{\begin{aligned} y_0 x_1 + x_0 y_1 &= \sqrt{x_0 + 1} \\ 2x_0 x_1 \quad y_1 &= x_0 \end{aligned}}$$

Using $x_0 = 3$, $y_0 = 9$ in the $O(\varepsilon)$ problem yields the linear problem

$$(2) \qquad 9x_1 + 3y_1 = 2$$

$$(3) \qquad 6x_1 - y_1 = 3$$

whose solution is $x_1 = \frac{11}{27}$, $y_1 = -\frac{5}{9}$ to conclude

$$x = 3 + \frac{11}{27}\varepsilon + O(\varepsilon^2)$$

$$y = 9 - \frac{5}{9}\varepsilon + O(\varepsilon^2)$$

Alternate Method (General Theory)

$$Df = \begin{bmatrix} y - \frac{\varepsilon}{2\sqrt{x+1}} & x \\ 2x - \varepsilon & -1 \end{bmatrix} \qquad \frac{\partial f}{\partial \varepsilon} = \begin{pmatrix} -\sqrt{1+x} \\ -x \end{pmatrix}$$

For the expansion $\bar{x} = \bar{x}_0 + \varepsilon \bar{x}_1 + O(\varepsilon^2)$ wher $\bar{x}_0 = (3, 9)^T$

$$(4) \quad Df(\bar{x}_0, 0) = \begin{bmatrix} 9 & 3 \\ 6 & -1 \end{bmatrix} \qquad \frac{\partial f}{\partial \varepsilon}(\bar{x}_0, 0) = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

Generally theory for $\bar{x}_1$ is

$$(5) \qquad Df(\bar{x}_0, 0)\, \bar{x}_1 = -\frac{\partial f}{\partial \varepsilon}(\bar{x}_0, 0)$$

Using (4) in (5) we see (5) is (2)-(3) above.

Perturbed Linear Systems. $f(x,\varepsilon) = A(\varepsilon)x - b(\varepsilon)$

Let $A(\varepsilon) \in \mathbb{R}^{n\times n}$, $b(\varepsilon) \in \mathbb{R}^n$ be regular in ε and assume $A(0)$ is invertible.
Seek a regular expansion

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

for

$$A(\varepsilon)\, x = b(\varepsilon)$$

we expand out A, x and b and collect like powers of ε.

$$(A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots)(x_0 + \varepsilon x_1 + \cdots) = (b_0 + \varepsilon b_1 + \cdots)$$

yields

$O(1)$ $$A_0\, x_0 = b_0$$

$O(\varepsilon)$ $$A_0\, x_1 = b_1 - A_1 x_0$$

$O(\varepsilon^2)$ $$A_0\, x_2 = b_2 - A_1 x_1 - A_2 x_0$$

Since A_0 invertible

$$x = A_0^{-1} b_0 + \varepsilon\left(A_0^{-1} b_1 - A_0^{-1} A_1 x_0\right) + \varepsilon^2\left(A_0^{-1} b_2 - A_0^{-1} A_1 x_1 - A_0^{-1} A_2 x_0\right) + \cdots$$

Collecting the indicated term we have a simple result:

$$x = \underbrace{A_0^{-1} b(\varepsilon)}_{\text{wiggle } b} - \varepsilon \underbrace{A_0^{-1} A_1 x(\varepsilon)}_{\text{leading effect of } A \text{ perturbation}} - \varepsilon^2 A_0^{-1} A_2 x(\varepsilon) + O(\varepsilon^3)$$

wiggle b

leading effect of A perturbation.

Nearly Singular Linear Systems

Suppose $A(\varepsilon) \in \mathbb{R}^{n\times n}$ and $b(\varepsilon) \in \mathbb{R}^n$ both have regular expansions in ε but $A(0) = A_0$ is not invertible. Then it is not necessarily true that

$$A(\varepsilon)\, x = b(\varepsilon)$$

has a regular solution x.

We illustrate this by two simple examples:

EXAMPLE

$$\begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad x = \begin{pmatrix} \varepsilon^{-1} \\ 2-\varepsilon^{-1} \end{pmatrix}$$

Clearly $A(0)$ is not invertible. Here $x(\varepsilon)$ has singular behavior in ε.

EXAMPLE

$$\begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3+\varepsilon \\ 2\varepsilon \end{bmatrix} \qquad x = \begin{pmatrix} 1+\varepsilon \\ 2 \end{pmatrix}$$

Same matrix $A(\varepsilon)$ and $b(\varepsilon)$ is regular in ε. In this case $x(\varepsilon)$ is regular in ε.

Perturbed Eigenvalue Problem

(1) $$A(\varepsilon)\,x = \lambda x \qquad A \in \mathbb{R}^{n\times n}$$

We assume coefficients $a_{ij}(\varepsilon)$ are smooth in ε. Consequently

(2) $$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots$$

One might expect $\lambda(\varepsilon)$ to be smooth but this is not always the case. A simple example is

$$A = \begin{bmatrix} 1 & 4 \\ \varepsilon & 1 \end{bmatrix} \qquad P(\lambda) = \det(A - \lambda I) = (\lambda - 1)^2 - 4\varepsilon$$

Here $\lambda(\varepsilon) = 1 \pm 2\sqrt{\varepsilon}$ which has no series expansion in ε and $\lambda'(0)$ undefined.

<u>Theorem</u>: If λ_0 is a <u>simple</u> eigenvalue of A_0 then $\forall \varepsilon \in N_r(0)$, certain λ_K

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots$$

<u>Pf</u> Define the characteristic polynomial $P(\lambda, \varepsilon) \equiv \det(A(\varepsilon) - \lambda I)$. Hypotheses imply $P(\lambda_0, 0) = 0$. And there is a polynomial $Q(\lambda)$ of degree $n-1$ such that

$$P(\lambda, 0) = (\lambda - \lambda_0)\,Q(\lambda)$$

$$P_\lambda(\lambda, 0) = Q(\lambda) + \cancelto{0}{(\lambda - \lambda_0)}\,Q'(\lambda)$$

$$P_\lambda(\lambda_0, 0) = Q(\lambda_0) \neq 0 \quad \text{since } \lambda_0 \text{ simple}$$

By implicit fn theorem $\lambda(\varepsilon) = \lambda_0 + \varepsilon\lambda_1 + \cdots$ □

Expansions for simple eigenvalues

(1) $$(A(\varepsilon) - \lambda I)x = 0$$

Substitute the expansions

(2) $$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots \qquad \phi \in \{A, x, \lambda\}$$

into equation (1) and collect powers of ε:

$O(1)$ $$(A_0 - \lambda_0 I)x_0 = 0$$

$O(\varepsilon)$ $$(A_0 - \lambda_0 I)x_1 = (\lambda_1 I - A_1)x_0$$

$O(\varepsilon^2)$ $$(A_0 - \lambda_0 I)x_2 = (\lambda_1 I - A_1)x_1 + (\lambda_2 I - A_2)x_0$$

Since λ_0 is simple $\dim E_{\lambda_0}(A_0) = 1$. Let x_0 be any eigenvector. By the fundamental theorem of linear algebra

$$\dim N((A_0 - \lambda_0 I)^T) = 1$$

Let $v \in N(A_0^T - \lambda_0 I)$. The $O(\varepsilon)$ problem has a soln only if (Fredholm Alternative)

$$v^T(\lambda_1 I - A_1)x_0 = 0$$

Solving for λ_1

$$\lambda_1 = \frac{v^T A_1 x_0}{v^T x_0}$$

Note that $v = x_0$ when A_0 is symmetric.

Regular expansions of definite integrals

By way of example we seek an approximation of

$$I(\varepsilon) = \int_a^b F(x,\varepsilon)\,dx \qquad 0<\varepsilon \ll 1$$

where $[a,b]$ closed and bounded and F is smooth.

EXAMPLE

$$I(\varepsilon) = \int_0^1 \frac{\cos(\varepsilon x)}{x^2+1}\,dx$$

For fixed x, $\cos(\varepsilon x) = 1 - \frac{1}{2!}\varepsilon^2 x^2 + \frac{1}{4!}\varepsilon^4 x^4 + O(\varepsilon^6)$

$$I(\varepsilon) = \int_0^1 \left(\frac{1}{(x^2+1)} - \frac{1}{2!}\frac{x^2}{(x^2+1)}\varepsilon^2 + \frac{1}{4!}\frac{x^4}{(x^2+1)}\varepsilon^4 + \cdots \right) dx$$

Integrate term by term to get

$$(1) \qquad I(\varepsilon) = \frac{\pi}{4} + \left(\frac{1}{2} - \frac{\pi}{8}\right)\varepsilon^2 + O(\varepsilon^4)$$

This integral can be expressed in terms of $Si(x)$ and $Ci(x)$ functions but such expressions have less value ... the $O(\varepsilon^2)$ term is very easy to understand.

Regularly perturbed IVP - Projectile

By way of example. Suppose a mass m is shot up from the earth's surface at velocity v_0. Gravity acts on the mass so that the IVP is

$$(1) \qquad \frac{d^2x}{d\tau^2} = -\frac{gR^2}{(x+R)^2}$$

$$(2) \qquad x(0) = 0$$

$$(3) \qquad x'(0) = v_0 \qquad \text{(initial velocity)}$$

where

$x(\tau)$ = altitude at time τ

R = earth's radius

$g = \dfrac{GM_e}{R^2}$ = surface grav. constant

Note that when x is small relative to R the eqn (1) may be approximated by

$$x'' = -g \qquad \text{(near surface)}$$

However, if v_0 is large a projectile may reach appreciable heights.

<u>Nondimensionalize</u>:

$$y = x/x^* \qquad t = \tau/\tau^*$$

where

$$x^* = \frac{v_0^2}{g} \qquad \tau^* = \frac{v_0}{g}$$

Nondimensionalized problem:

$$(4) \qquad y'' = -\frac{1}{(1+\varepsilon y)^2} = f(y,\varepsilon)$$

$$(5) \qquad y(0) = 0$$

$$(6) \qquad y'(0) = 1$$

where

$$\varepsilon = \frac{v_0^2}{Rg} \ll 1$$

on account of earth's radius R being so large

The solution $y = y(t,\varepsilon)$ of (4) depends smoothly on ε but we postpone any further discussion. We expand

$$(7) \qquad y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \cdots$$

Use the binomial theorem to expand $f(y,\varepsilon)$

$$(1+\varepsilon y)^{-2} = 1 - 2\varepsilon y + O(\varepsilon^2)$$

Expanded, eqns (4) and (6) are

$$y_0'' + \varepsilon y_1'' = -1 + 2\varepsilon y_0 + O(\varepsilon^2)$$

$$y_0'(0) + \varepsilon y_1'(0) = 1 + O(\varepsilon^2)$$

Matching powers of ε in both - the ODE and Initial Conditions

$O(1)$ $\quad y_0'' = -1 \qquad y_0(0) = 0 \qquad y_0'(0) = 1$

$O(\varepsilon)$ $\quad y_1'' = 2y_0 \qquad y_1(0) = 0 \qquad y_1'(0) = 0$

The solution to this sequence of IVP is

$$y_0(t) = -\tfrac{1}{2}t^2 + t$$

$$y_1(t) = -\tfrac{1}{12}t^4 + \tfrac{1}{3}t^3$$

Such solns can be used to answer useful question. For instance, if v_0 is large enough the flight time $T(\varepsilon)$ should increase. Let

$$T(\varepsilon) = T_0 + \varepsilon T_1 + O(\varepsilon^2)$$

Then

$$y_0(T(\varepsilon)) + \varepsilon y_1(T(\varepsilon)) + O(\varepsilon^2) = 0$$

Expanding and collecting in ε^n

$$\underset{\uparrow}{y_0(T_0)} + (y_0'(T_0)\underset{\uparrow}{T_1} + y_1(T_0))\varepsilon = O(\varepsilon^2)$$

Indicated terms vanish ultimately resulting in

$$T = 2 + \underbrace{\tfrac{4}{3}\varepsilon}_{\text{correction for increased flight time.}} + O(\varepsilon^2)$$

Regularily Perturbed 2-point BVP

Define the following operator and domains

$$L(u) \equiv -(p(x)u')' + q(x)u$$

$$D(L) = \{ u \in C^2[a,b] : B_1(u) = \alpha, B_2(u) = \beta \}$$

$$D_0(L) = \{ u \in C^2[a,b] : B_1(u) = B_2(u) = 0 \}$$

where $B_K(u)$ are boundary operators. For instance

$$B_K(u) = c_1 u(a) + c_2 u'(a)$$

We seek a regular expansion for the solution $u(x,\varepsilon)$ to the weakly nonlinear BVP

$$\boxed{\begin{gathered} Lu - \lambda u = \varepsilon f(u, u', x) \\ u \in D(L) \end{gathered}}$$

where $0 < \varepsilon \ll 1$. Letting

$$u(x,\varepsilon) = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^2)$$

one finds

$O(1)$ $\quad Lu_0 - \lambda u_0 = 0 \qquad u_0 \in D(L)$

$O(\varepsilon)$ $\quad Lu_1 - \lambda u_1 = f(u_0, u_0', x) \qquad u_1 \in D_0(L)$

Note u_0 and u_1 satisfy different B.C.

Though $\lambda = 0$ is possible its inclusion can illustrate some issues

<u>λ an eigenvalue of L</u>

This would mean $N(L-\lambda I) \neq 0$ in which case the $O(1)$ problem

$$O(1) \qquad Lu_0 - \lambda u_0 = 0 \qquad u_0 \in D(L)$$

would not have a unique solution. Even worse the problem may not have a regular solution at all. Consider the $O(\varepsilon)$ problem

$$O(\varepsilon) \qquad Lu_1 - \lambda u_1 = f \qquad u_1 \in D_0(L)$$

where $f = f(u_0, u_0', x)$. Noting L is self adjoint and $N(L-\lambda I) \neq 0$ the Fredholm Alternative implies the $O(\varepsilon)$ problem will have a solution only if

$$\langle v, f \rangle = 0 \qquad \forall v \in N(L-\lambda I)$$

As to whether this is satisfied depends largely on f.

<u>λ not an eigenvalue</u>

Since $N(L-\lambda I) = 0$ the solutions of all $O(\varepsilon^n)$ problems is unique for $n = 0, 1, 2 \ldots$

Remarks on Green's Functions

Suppose λ is not an eigenvalue of L, and $u_0(x)$ is the solution of

$$Lu_0 - \lambda u_0 = 0 \qquad\qquad u_0 \in D(L)$$

This must have nonhomogeneous B.C. otherwise $u_0(x) \equiv 0$.

The $O(\varepsilon)$ problem

$$Lu_1 - \lambda u_1 = f \qquad\qquad u_1 \in D_0(L)$$

has a Green's function associated with it in part since the B.C. are homogeneous Consequently

$$u_1 = \langle G_\lambda, f \rangle$$

or more explicitly

$$u_1(x) = \int_a^b G_\lambda(x,\xi)\, f(u_0(\xi), u_0'(\xi), \xi)\, d\xi$$

Expressions for higher order expansions, i.e., u_2, are more easily written out using such inner products.

EXAMPLE

$$\boxed{\begin{array}{ll} u'' + \lambda^2 u = 3\varepsilon u u' & \\ u(0) = 0 & u(1) = 1 \end{array}}$$

The leading $O(1)$ problem is

$$u_0'' + \lambda^2 u_0 = 0 \qquad u_0(0) = 0 \quad u_0(1) = 1$$

and so long as $\lambda \neq \lambda_n = n\pi$ (eigenvalues)

(1) $$u_0(x) = \frac{\sin(\lambda x)}{\sin(\lambda)}$$

Then the $O(\varepsilon)$ problem is

$$u_1'' + \lambda^2 u_1 = 3 u_0 u_0'$$

$$u_1(0) = 0 \quad u_1(1) = 0 \qquad \text{(homogenous)}$$

After considerable calculations

(2) $$u_1(x) = \frac{\sin(\lambda x)(\cos\lambda - \cos(\lambda x))}{\lambda \sin^2\lambda}$$

can't be zero.

Note that $u_1(x)$ is also undefined at eigenvalues of L.

Final Remark If a Green's fn for L on $D_0(L)$ is known

$$u(x,\varepsilon) = u_0(x) + \varepsilon \int_a^b G(x,\xi) f(u_0(\xi), u_0'(\xi), \xi)\, d\xi + O(\varepsilon^2)$$

Perturbed eigenvalue problems

Let $L : D(L) \to H$ be some linear operator defined on $D(L) \subset H$ where H is a Hilbert space. Further let $f : H \to H$. Seek higher order approximations to eigenvalue $\lambda(\varepsilon)$ of

$$Lu = \lambda u + \varepsilon f \qquad u \in D(L)$$

For $\varepsilon = 0$ we assume

$$Lu_0 = \lambda_0 u_0 \qquad u_0 \in D(L)$$

and $\dim N(L - \lambda_0 I) = 1$ so that λ_0 is simple. Assuming

$$u(x,\varepsilon) = u_0 + \varepsilon u_1 + O(\varepsilon^2) \qquad *$$

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + O(\varepsilon^2)$$

one readily finds

$O(1)$ $\qquad Lu_0 = \lambda_0 u_0$

$O(\varepsilon)$ $\qquad Lu_1 = \lambda_0 u_1 + \lambda_1 u_0 + f(u_0)$

Let $v \in N(L - \lambda_0 I)$. The Fredholm Alternative $\Rightarrow$

$$\langle v, \lambda_1 u_0 + f(u_0) \rangle = 0 \qquad (v = u_0 \text{ w.l.o.g.})$$

Solving for

$$\lambda_1 = - \frac{\langle u_0, f(u_0) \rangle}{\| u_0 \|^2}$$

* smoothness depends on L, $D(L)$ and H.

Perturbed Domain Problems (Regular)

Let $u = u(x)$ be the solution of

$$Lu = f(x) \qquad x \in \Omega_\varepsilon$$

$$u = g(x, \varepsilon) \qquad x \in \partial\Omega_\varepsilon$$

where Ω_ε is a perturbed domain of Ω_0

By way of example we show how to construct a regular expansion for $u(x, \varepsilon)$.

EXAMPLE $\quad \Omega_\varepsilon = [0, \frac{\pi}{2} + \varepsilon]$

$$(1) \qquad u'' + u = 1$$

$$(2) \qquad u(0, \varepsilon) = 0$$

$$(3) \qquad u(\tfrac{\pi}{2} + \varepsilon, \varepsilon) = 1$$

Let

$$u(x, \varepsilon) = u_0(x) + \varepsilon u_1(x) + \cdots$$

Results in the following expansion for the right boundary

$$u(\tfrac{\pi}{2} + \varepsilon, \varepsilon) = u_0(\tfrac{\pi}{2} + \varepsilon) + \varepsilon u_1(\tfrac{\pi}{2} + \varepsilon) + O(\varepsilon^2)$$

$$(4) \quad u(\tfrac{\pi}{2} + \varepsilon, \varepsilon) = u_0(\tfrac{\pi}{2}) + \left(u_0'(\tfrac{\pi}{2}) + u_1(\tfrac{\pi}{2})\right)\varepsilon + \cdots$$

Collecting $O(1)$ and $O(\varepsilon)$ terms

$O(1)$ $\quad u_0'' + u_0 = 1 \qquad u_0(0) = 0 \quad u_0\left(\frac{\pi}{2}\right) = 1$

$O(\varepsilon)$ $\quad u_1'' + u_1 = 0 \qquad u_1(0) = 0 \quad u_1\left(\frac{\pi}{2}\right) = -u_0'\left(\frac{\pi}{2}\right)$

Solving these in succesion

$$u_0(x) = 1 - \cos x$$

$$u_1(x) = -\sin x$$

which yields

$$(1) \qquad u(x,\varepsilon) = (1 - \cos x) - \varepsilon \sin x + O(\varepsilon^2)$$

and seen to be an expansion of exact soln

$$u(x,\varepsilon) = (1-\cos x) - \tan\varepsilon \sin x$$

<u>Amplitude Phase form of solution</u>

Seek a solution of the form $u = A\cos(x+\phi) + 1$

$$A\cos(x+\phi) = \underline{A\cos\phi}\cos x \underline{- A\sin\phi}\sin x$$

match terms to (1) above

$$A\cos\phi = 1 \qquad -A\sin\phi = \varepsilon$$

yields $A = \sqrt{1+\varepsilon^2}$ and $\phi = \arctan(-\varepsilon) = -\varepsilon + O(\varepsilon^3)$:

$$u(x,\varepsilon) = 1 - \sqrt{1+\varepsilon^2}\cos(x-\varepsilon) + O(\varepsilon^2)$$

EXAMPLE **Expansions need not be regular**

$$(1) \qquad u'' + u = 1 \qquad x \in \Omega_\varepsilon = [0, \pi + \varepsilon]$$

$$(2) \qquad u(0) = 0 \quad , \quad u(\pi + \varepsilon) = 0$$

Here the leading $\varepsilon = 0$ problem does not have a unique solution

$$u_0(x) = 1 + c \sin x \qquad \forall c \in \mathbb{R}$$

However (1)-(2) does have a unique solution for all $\varepsilon > 0$

$$u(x, \varepsilon) = (1 - \cos x) + \frac{(1 + \cos \varepsilon) \sin x}{\sin \varepsilon}$$

As $\varepsilon \to 0^+$ this has a singular expansion

$$u(x, \varepsilon) \sim \underbrace{\frac{2 \sin x}{\varepsilon}}_{\text{singular}} + (1 - \cos x) + O(\varepsilon)$$

Remarks on perturbed boundaries

Expansions need not be regular

(1) $u'' + u = 1$

(2) $u(0) = 0 \qquad u(\pi + \varepsilon) = 0$

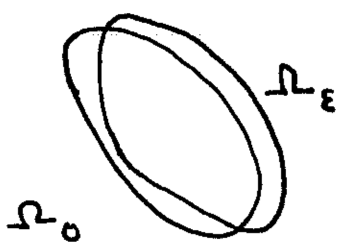
has the singular solution

$$u(x,\varepsilon) = (1 - \cos x) + \frac{\sin x\,(1 + \cos\varepsilon)}{\sin\varepsilon}$$

$$u(x,\varepsilon) \sim \underbrace{\frac{2\sin x}{\varepsilon}}_{\text{singular}} + (1 - \cos x) + O(\varepsilon)$$

Domains $\Omega_\varepsilon \subset \mathbb{R}^n$

May occur for PDE problems



Stresses induced by small changes to domain $\Omega_\varepsilon \subset \mathbb{R}^2$

$$\nabla^4 \phi = 0$$

is biharmonic equation.

The Airy stress function $\phi(x,y)$ generates

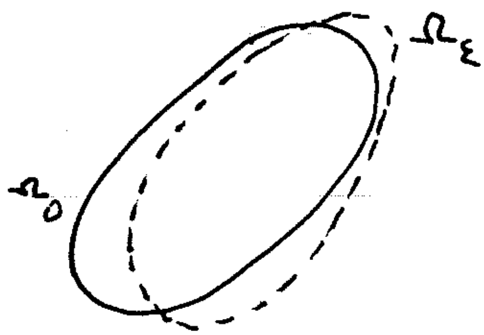
$$\left.\begin{aligned} \sigma_x &= \phi_{xx} \\ \sigma_y &= \phi_{yy} \end{aligned}\right\} \text{normal stresses}$$

$$\left.\tau_{xy} = -\phi_{xy}\right\} \text{shear stresses.}$$

Perturbed Domains $\Omega_\varepsilon \subset \mathbb{R}^n$

A simple relevant example involves the changes of stress induced by a small shape change

Let Ω_ε be a slight deformation of a plate $\Omega_0 \subset \mathbb{R}^2$.



The stresses in the plate are found by solving the eqn:

$$\nabla^4 \phi = 0$$

Biharmonic equation

Here

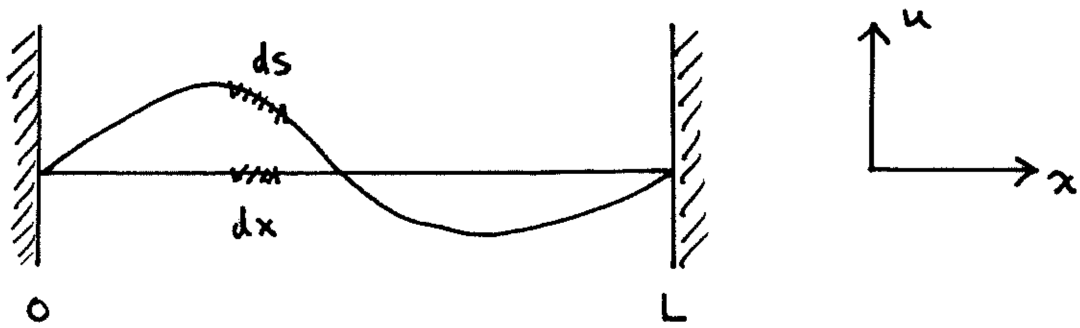
$$\nabla^4 \phi = \nabla^2(\nabla^2 \phi) = \phi_{xxxx} + 2\phi_{xxyy} + \phi_{yyyy}$$

Once solved

$$\left.\begin{aligned} \sigma_x &= \phi_{xx} \\ \sigma_y &= \phi_{yy} \end{aligned}\right\} \text{normal stresses}$$

$$\left.\tau_{xy} = -\phi_{xy}\right\} \text{shear stresses}$$

Small Amplitude Waves



A sting vibrating having amplitude $u(x,t)$

Law of mass action requires the following action be extremized

$$J(u) = \int_{t_1}^{t_2} \int_0^L \Big(\underbrace{\tfrac{1}{2}\sigma u_t^2}_{T} - \underbrace{\tau\sqrt{1+u_x^2}}_{V} \Big)\, dx\, dt$$

where $[\sigma] = kg/m$ and τ = tension (force)

$$ds = \sqrt{1+u_x^2}\; dx$$

so the second term in J is the total elastic potential. Has the form

$$(1) \qquad J(u) = \iint_\Omega L(u, u_x, u_y)\, dA$$

with Lagrangian $(t \leftrightarrow y)$

$$(2) \qquad L = \tfrac{1}{2}\sigma u_y^2 - \tau\sqrt{1+u_x^2}$$

From general theory extrema of (1) satisfy

$$\frac{\partial L}{\partial u} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right)$$

when applied to (2) we obtain

$$\boxed{u_{tt} = D\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)} \tag{3}$$

where we assume $D \equiv \tau/\sigma$ is constant.

Clearly (3) is too difficult to solve directly so instead we seek a small amplitude approximation of the form

$$u(x,t,\varepsilon) = \varepsilon u_0(x,t) + \varepsilon^2 u_1(x,t) + O(\varepsilon^3) \tag{4}$$

After using (4) in (3) and expanding in ε

$O(1)$ $\qquad u_{0,tt} = D u_{0,xx}$

$O(\varepsilon)$ $\qquad u_{1,tt} = D u_{1,xx}$

$O(\varepsilon^2)$ $\qquad u_{2,tt} = D u_{2,xx} - \frac{3}{2} u_{0,x}^2 u_{0,xx}$

The leading problem is <u>the wave equation</u>!

EXAMPLE

$$u_{tt} = \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2}}\right) \qquad \text{(PDE)}$$

$$u(0,t) = u(\pi,t) = 0 \qquad \text{(BC)}$$

$$u(x,0) = 0 \qquad \text{(IC)}$$

$$u_t(x,0) = \varepsilon \sin x \qquad \text{(IC)}$$

Solution of $O(1)$ problem

$$\frac{\partial^2 u_0}{\partial t^2} = \frac{\partial^2 u_0}{\partial x^2}$$

satisfies same homogeneous (BC) and (IC)

$$u_0(x,0) = 0 \qquad u_{0,t}(x,0) = \sin x$$

The solution is

$$u_0(x,t) = \sin x \sin t$$

Solution of $O(\varepsilon)$ problem

$$u_{1,tt} = u_{1,xx}$$

$$u_1(0,t) = u_1(\pi,t) = 0$$

$$u_1(x,0) = u_{1,t}(x,0) = 0$$

has the unique solution $u_1(x,t) \equiv 0$!!

Solution of $O(\varepsilon^2)$ problem

For notational simplicity $v(x,t) = u_2(x,t)$

$$(1) \qquad v_{tt} = v_{xx} - \frac{3}{2} u_{0,x}^2 \, u_{0,xx}$$

$$v_{tt} = v_{xx} + f(x,t)$$

where

$$f(x,t) = -\frac{3}{2} \sin^3 t \, (\sin x \cos^2 x)$$

Ultimately we seek a Fourier series solution to (1) in terms of the eigenfunctions $\phi_n(x) = \sin(nx)$.

$$(2) \qquad v(x,t) = \sum_{n=1}^{\infty} a_n(t) \, \phi_n(x)$$

Toward this end we need the Fovier expansion of $f(x,t)$ which can be found using trig idents: $\sin x \cos^2 x = \frac{1}{2} \sin(2x) \cos x$.

$$(3) \qquad f(x,t) = \frac{3}{8} \sin^3 t \, (\underset{\substack{\uparrow \\ n=1}}{\sin x} + \underset{\substack{\uparrow \\ n=3}}{\sin(3x)})$$

Has only two nontrivial modes.

The expansion (2) satisfies the boundary conditions $v(0,t) = v(\pi, t) = 0 \ \forall t$.
The conditions

$$v(x,0) = 0 \qquad v_t(x,0) = 0$$

imply $a_n(0) = a_n'(0) = 0$ for all n. Substitute (2) into and use (3):

$$\sum_{n=1}^{\infty} a_n'' \phi_n(x) = \sum_{n=1}^{\infty} -n^2 a_n \phi_n(x) + f(x,t)$$

Orthogonality of $\{\phi_n\}$ implies for $n \neq 1, 3$

$$\left\| \begin{array}{l} a_n'' + n^2 a_n = 0 \\ a_n(0) = a_n'(0) = 0 \end{array} \right\| \qquad n \neq 1, 3$$

whose solution is $a_n(t) \equiv 0$. For $n = 1, 3$

$$\left\| \begin{array}{l} a_n'' + n^2 a_n = \frac{3}{8} \sin^3 t \\ a_n(0) = a_n'(0) = 0 \end{array} \right\| \qquad n = 1, 3$$

Both of these modes exhibit resonance.
For instance

$$a_1(t) = a_1^{(osc)}(t) \underbrace{- \frac{9}{64} t \cos t}_{\text{resonant growth of } n=1 \text{ mode.}}$$

where $a_1^{(osc)}(t)$ is a periodic function. Since $u = \varepsilon u_0 + \varepsilon^3 u_2 + O(\varepsilon^4)$ one must have $t = O(\varepsilon^{-2})$ time before they are observed in εu_0 term.