SYMMETRY BREAKING HOMOCLINIC BIFURCATIONS IN DIFFUSIVELY COUPLED EQUATIONS*

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Abstract. In this study we examine a symmetry breaking bifurcation of homoclinic orbits in diffusively coupled ordinary differential equations. We prove that asymmetric homoclinic orbits can bifurcate from a symmetric homoclinic orbit when the equilibria to which the latter is homoclinic undergoes a pitchfork bifurcation. A condition which defines the direction of the bifurcation in a parameter space is given. All hypotheses of the main theorem are verified for a diffusively coupled logistic system and the twistedness of the bifurcating homoclinic orbits is computed for a range of coupling strengths.

Key words. homoclinic bifurcations, twistedness, coupled oscillators

AMS(MOS) subject classifications. 34C23, 92C10,34C15, 34C37, 92B99

1. Introduction. Many models of biochemical reactions and other biological phemonema can be represented by a system of ordinary differential equations known to possess stable limit cycles. Typically these periodic solutions exist in some parameter range and can appear or disappear as a parameter varies. One such bifurcation is the well studied Hopf bifurcation where a periodic solution bifurcates from an equilibrium as the parameter passes through a critical value.

In other instances, periodic motion can result from homoclinic bifurcations. Namely, if the system has a non-degenerate homoclinic orbit at a certain parameter value, then periodic orbits can bifurcate from this homoclinic orbit as the parameter is varied from this value. Furthermore, under certain conditions, these bifurcating periodic orbits are stable for parameter values near the bifurcation point. Examples of models exhibiting such bifurcations are numerous - especially in excitable biological systems. A few such examples include the model of mechano-electrical excitability of muscle cells [14] and the fast subsystems of models exhibiting bursting [17, 16]. The focus of this study is to identify a specific type of symmetry breaking bifurcation of homoclinic orbits induced by coupling.

We restrict our attention to the two component system of ordinary differential equations

$$\dot{x} = f(x,\mu), \qquad x \in \mathbb{R}^2, \quad \mu \in J \subset \mathbb{R} \tag{1.1}$$

where $(\dot{\ }) = \frac{d}{dt}(\ )$, μ is a parameter, and f is a smooth function from $\mathbb{R}^2 \times J$ into $\mathbb{R}^2$ (at least C^4). We assume $x = 0$ is a hyperbolic equilibrium for all $\mu \in J$ and that there is a homoclinic orbit $x = x_h(t)$ asymptotic to $x = 0$ when $\mu = 0$, that is

$$\dot{x}_h(t) = f(x_h(t), 0), \qquad \lim_{t\to\pm\infty} x_h(t) = 0. \tag{1.2}$$

We also assume a nondegeneracy condition for the homoclinic orbit so that it breaks as μ passes through $\mu = 0$.

Under these assumptions we then consider the diffusively coupled system

$$\begin{aligned} \dot{x}^1 &= f(x^1,\mu) + \nu D(x^2 - x^1), \\ \dot{x}^2 &= f(x^2,\mu) + \nu D(x^1 - x^2) \end{aligned} \tag{1.3}$$

* This work was supported by the National Science Foundation grants OSR-93-50-546, DMS-94-04-521, DMS-94-04-160 and by the Ryukoku Joint Research Center for Science and Technology

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where ν is a positive parameter expressing the strength of the coupling and D is a diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad d_j \geq 0, \quad d_1 + d_2 > 0.$$

Throughout the manuscript, we shall say a solution of (1.3) is "homogeneous" or "symmetric" if it can be written in the form $(x^1, x^2) = (\tilde{x}(t), \tilde{x}(t))$ where $\tilde{x}(t)$ is any solution of (1.1). Note, symmetric solutions of (1.3) are invariant under reflection. Moreover, we easily see that (1.3) has a homogeneous (or symmetric) homoclinic solution $(x^1, x^2) = (x_h(t), x_h(t))$ asymptotic to the equilibrium $(0, 0)$ when $\mu = 0$.

Next, we assume that a symmetry breaking bifurcation of equilibria from the unstable equilibrium $(x^1, x^2) = (0, 0)$ occurs at a certain value of ν, say $\nu = \nu_h$. Explicit conditions for such a bifurcation of equilibria can easily be realized and are discussed in more detail in section 3.

This last assumption and the existence of a symmetric homoclinic orbit at $\mu = 0$ naturally leads to the following question: can a symmetry breaking homoclinic bifurcation be induced by the bifurcation of the equilibria? Namely, when the pair of equilibria bifurcate from $(x^1, x^2) = (0, 0)$, can asymmetric homoclinic orbits (asymptotic to the bifurcating equilibria) bifurcate from the symmetric homoclinic orbit? And, if so, where are these asymmetric homoclinic solutions located in the (μ, ν) parameter plane.

In this article we answer these questions by examining these bifurcations near $(\mu, \nu) = (0, \nu_h)$. From a geometric characterization of the symmetric homoclinic orbit there are two generic cases. We show that if the homoclinic orbit is "nontwisted", a pair of asymmetric homoclinic orbits bifurcates from the symmetric one. Conversely, if the homoclinic orbit is "twisted", a pair of asymmetric heteroclinic orbits bifurcate instead. In either case, these bifurcating solutions are shown to persist (locally) on a smooth curve eminating from $(0, \nu_h)$ in (μ, ν)-plane. Here the adopted definitions of "twisted" and "nontwisted" (see Fig 1) are those found in the work by Lin [12] on a particular reaction-diffusion system.

In our main theorem, an explicit condition showing the direction of the bifurcation curve in the (μ, ν)-plane (eminating from $(\mu, \nu) = (0, \nu_h)$) is determined by solving a bifurcation equation. The bifurcation curve appears in the direction $\mu > 0$ (or $\mu < 0$) depending on the sign of coefficients derived from the bifurcation equation. This fact suggests an interesting dynamic structure of (1.3) provided a stable periodic solution $x_p(t)$ of (1.1) bifurcates from the homoclinic solution $x_h(t)$ in a direction $\mu > 0$. Actually, some of the work on the symmetry breaking bifurcation from a homogeneous homoclinic orbit by Lin [12] can be applied to our system to show that, under the hypotheses of our theorem and additional conditions on the bifurcation of $x_p(t)$, there exist a pair of asymmetric periodic solutions (or a symmetric double periodic solution) bifurcating from the homogeneous homoclinic orbit. Although Lin's results do not show the location of the periodic solutions in the (μ, ν)-plane, they do suggest a connection between the homoclinic/heteroclinic bifurcations and periodic solutions bifurcating from these homoclinic/heteroclinic orbits.

The results presented herein may also be contrasted with those derived in [1]. In [1], bifurcating homoclinic solutions were shown to exist in the vicinity of twist points (see Theorem 6.1, pg. 213 [1]). In contrast, we shown that bifurcating homoclinic solutions emerge near bifurcating equlibria, not necessarily near twist points. This fact was also demonstrated in an example where our hypotheses can be verified explicitly.

We emphasize that even though the system (1.1) may have no periodic solution for $\mu < 0$, the diffusively coupled system (1.3) can have a periodic motion for this region. The bifurcation from the asymmetric homoclinic orbit (resp. heteroclinic orbit) to an asymmetric periodic solution (resp. symmetric double periodic solution) will be studied in the future.

Our approach to obtain the bifurcation equation is straightforward. We seek a condition for which a 1-dimensional unstable manifold eminating from an asymmetric equilibrium enters the center-stable manifold of $(x^1, x^2) = (0, 0)$. This condition is obtained by using a Melnikov function, and ultimately leads to a bifurcation equation. A direct computation of the leading-order solution is key to determining which of the asymmetric homoclinic or heteroclinic orbits appear. Additional computations involving this solution are then used to determine the direction of the bifurcating curve in the (μ, ν)-plane.

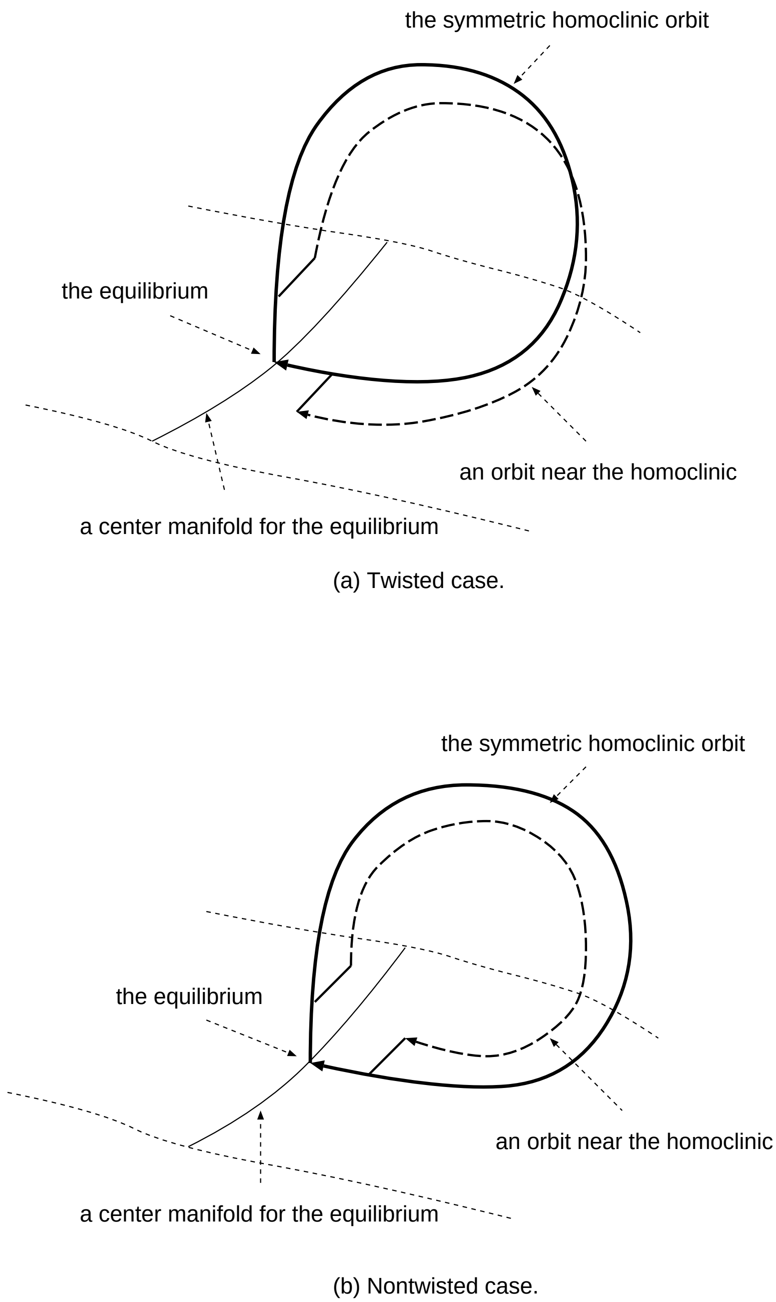

FIG. 1. *Twisted and nontwisted symmetric homoclinic orbits*

We remark that although our criteria for finding the direction of the bifurcation of the asymmetric homoclinics from the homogeneous one are explicitly given, they are not so simple to verify for any specific system (1.1) because some global information for the dynamics is involved. Therefore, for most cases, we need numerical computations to verify the direction. Fortunately, there is a specific model for which the directions and assumptions can be verified explicitly. This we present in the last section.

We start by introducing the assumptions for the main theorem. Let the matrix

$$A := \frac{\partial f}{\partial x}(0,0) \quad ,$$

have eigenvalues $\{\alpha_1, -\alpha_2\}$ and the corresponding eigenfuctions $\{\zeta_1, \zeta_2\}$, namely

$$A\zeta_j = (-1)^{j-1}\alpha_j\zeta_j \quad , \qquad j = 1, 2.$$

We assume that

$$\alpha_2 \quad , \quad \alpha_1 > 0 \quad . \tag{1.4}$$

The linearization of (1.1) about $x_h(t)$ is given by

$$\dot{p} = A(t)p, \qquad A(t) := \frac{\partial f}{\partial x}(x_h(t), 0) \quad , \tag{1.5}$$

which has a bounded solution $p_1 = \dot{x}_h(t)$ over $\mathbb{R}$. Consequently, the adjoint equation

$$\dot{q} = -A(t)^T q \tag{1.6}$$

also has a bounded solution on $\mathbb{R}$.

Let α_0 and $-\alpha_3$ be the eigenvalues of $A - 2\nu D$ and assume

$$\alpha_0 = 0, \quad \alpha_3 > 0 \quad \text{for} \quad \nu = \nu_h. \tag{1.7}$$

Furthermore, let ζ_0 (resp. ζ_0^*) be an eigenvector corresponding to the zero eigenvalue of $A - 2\nu_h D$ (resp. $(A - 2\nu_h D)^T$). We will show (in section 3) that under the assumption

$$\zeta_0^{*T} D\zeta_0 > 0 \tag{1.8}$$

asymmetric equlibria of (1.3) bifurcate from $(x^1, x^2) = (0, 0)$ as ν passes through ν_h with $\mu = 0$ fixed.

Since $A(t) \to A$ as $t \to \pm\infty$, the linear equation

$$\dot{y} = (A(t) - 2\nu_h D)y \tag{1.9}$$

has a solution $p_0(t)$ with asymptotics

$$\lim_{t\to-\infty} p_0(t) = \zeta_0, \qquad \lim_{t\to+\infty} p_0(t) = \hat{c}\ \zeta_0, \tag{1.10}$$

where $\hat{c}$ is a constant (which could be zero).

Now, let $q_2(t)$ be a bounded solution of the adjoint equation (1.6) satisfying

$$\lim_{t\to\infty} q_2(-t)^T \dot{x}_h(t) e^{2\alpha_2 t} > 0. \tag{1.11}$$

and assume

$$h_1 = \int_{-\infty}^{\infty} q_2(t)^T \partial_\mu f(x_h(t), 0)dt \ \neq \ 0. \tag{1.12}$$

We note that if the sign of (1.12) is negative and $\alpha_2 > \alpha_1$, then stable periodic solutions of (1.1) bifurcate from the homoclinic solution $x_h(t)$ in the direction $\mu > 0$ (see for instance [3], Theorem

10.3.2, p.364, Theorem 11.3.3, p. 390 and Corollary 11.3.7, p392). Lastly, we denote the expansion of $f(x, \mu)$ about $(x_0, 0)$ as

$$f(x_0 + x, \mu) = \partial_x f(x_0, 0)x + \partial_\mu f(0, 0)\mu + \frac{1}{2}\partial_x^2 f(x_0, 0) \circ (x, x) + \cdots.$$

We are now in a position to state the main theorem.

THEOREM 1.1. *Let $x_h(t)$ be a homoclinic solution of (1.1) asymptotic to $x = 0$ when $\mu = 0$. In addition to assuming the conditions (1.4), (1.7), (1.8) and (1.12), assume that the pitchfork bifurcation of equilibria of (1.3) from $(x^1, x^2) = (0, 0)$ occurs in a direction $\nu < \nu_h$ when $\mu = 0$. Namely, if the bifurcating asymmetric equilibria $E_1 = E_1(\epsilon)$ and $E_2 = E_2(\epsilon)$ are parametrized by a small parameter $\epsilon > 0$, then they exist for $\nu = \nu_h + \hat{\nu}(\epsilon)$ with $(d\hat{\nu}/d\epsilon)(0) = 0, (d^2\hat{\nu}/d\epsilon^2)(0) < 0$. Then*

(i) if $\hat{c} > 0$, there is a bifurcation curve $(\mu(\epsilon), \nu(\epsilon))$ for which (1.3) possesses a pair of asymmetric homoclinic orbits asymptotic to E_1 and E_2, respectively,

(ii) if $\hat{c} < 0$, there is a bifurcation curve $(\mu(\epsilon), \nu(\epsilon))$ for which (1.3) possesses a pair of asymmetric heteroclinic orbits connecting E_1 with E_2 and E_2 with E_1, respectively.

In either case, $(\mu(\epsilon), \nu(\epsilon))$ is a C^2-function in $\epsilon > 0$ with $(\mu(0), \nu(0)) = (0, \nu_h)$. Moreover $\mu(\epsilon)$ solves a bifurcation equation which has the expansion

$$H(\mu, \epsilon) = h_1\mu + \frac{1}{2}h_2\epsilon^2 + h.o.t = 0 \tag{1.13}$$

where

$$h_1 := \int_{-\infty}^{\infty} q_2(t)^T \partial_\mu f(x_h(t), 0)dt \tag{1.14}$$

$$h_2 := \int_{-\infty}^{\infty} q_2(t)^T \partial_x^2 f(x_h(t), 0) \circ (p_0(t), p_0(t))dt \tag{1.15}$$

Hence when $h_1h_2 < 0$ (resp. > 0), $\mu(\epsilon) > 0$ (resp. < 0) for sufficiently small $\epsilon > 0$.

We remark that in [12], the conditions $\hat{c} > 0$ and $\hat{c} < 0$ define the 'nontwisted' and 'twisted' cases, respectively. As mentioned previously, if $\alpha_2 > \alpha_1$, by Theorem 3.4 in [12] the nontwisted case induces a bifurcation of a pair of asymmetric periodic solutions whereas the twisted case induces a bifurcation of a double periodic solution.

The proof of the theorem is laid out in five parts. First, we examine the linearization of (1.1) about $x_h(t)$ deducing the relevant asymptotic behaviors of $q_2(t)$. The pitchfork of equilibria E_1 and E_2 are then examined in section 3. An exponential dichotomy for the linearization of the coupled system about the symmetric homoclinic solution is then found in section 4 and is subsequently used in section 5 to identify the bifurcation equation for the bifurcating homoclinic/heteroclinic solutions. This equation is then examined in detail in section 6, where the conclusions (i) and (ii) of the theorem are proven. We conclude the article by introducing a model system for which all the hypothesis of the theorem can be verified explicitly and then compute the twistedness of the bifurcating homoclinic orbits for a range of coupling strengths.

2. Symmetric homoclinic orbit.

In this section, we identify properties of the homoclinic orbit $x = x_h(t)$ of (1.1) which will subsequently be used to examine perturbations of the symmetric homoclinic solution of (1.3).

Under the assumption that $x_h(t)$ is homoclinic to the hyperbolic equilibrium $x = 0$, (1.5) admits an exponential dichotomy on each half line $\mathbb{R}_+$ or $\mathbb{R}_-$. This fact can be proved by some properties of the dichotomy (for instance see [7], § 2, 3). Here, however, we establish the dichotomy explicitly by examing the asymptotic behavior of solutions to (1.5) and the corresponding adjoint problem (1.6). Then the exponential dichotomy and the implicit function theorem yield a Melnikov function whose derivative determines how the homoclinic orbit of (1.1) breaks as μ varies. Lastly, we show that this Melnikov function can be expressed explicitly in terms of $x = x_h(t)$.

Since $x_h(t)$ satisfies (1.2) and the equilibrium $x = 0$ is hyperbolic, there are non-zero constants c_1 and c_2 such that

$$\lim_{t\to-\infty} x_h(t)e^{-\alpha_1 t} = c_1\zeta_1 \quad , \quad \lim_{t\to\infty} x_h(t)e^{\alpha_2 t} = c_2\zeta_2 \quad . \tag{2.1}$$

Let $\Phi_0(t)$ be a fundamental solution matrix of (1.5) written as

$$\Phi_0(t) = (p_1(t) \;\; p_2(t)) \quad . \tag{2.2}$$

We choose as the bounded solution

$$p_1(t) = \dot{x}_h(t) \quad . \tag{2.3}$$

Furthermore, by appropriately scaling ζ_i $(i = 1, 2)$ we may assume without any loss of generality that

$$\lim_{t\to-\infty} p_1(t)e^{-\alpha_1 t} = \zeta_1, \qquad \lim_{t\to\infty} p_1(t)e^{\alpha_2 t} = \zeta_2 \qquad . \tag{2.4}$$

Similarly, the unbounded solution $p_2(t)$ can be chosen with the asymptotic behaviors

$$\lim_{t\to-\infty} p_2(t)e^{\alpha_2 t} = \zeta_2, \qquad \lim_{t\to\infty} p_2(t)e^{-\alpha_1 t} = c\zeta_1 \qquad , \tag{2.5}$$

for some non-zero constant c.

Next, we consider the adjoint equation (1.6) whose fundamental solution matrix is given by

$$(\Phi_0(t)^{-1})^T = (q_1(t) \;\; q_2(t)) \quad . \tag{2.6}$$

Clearly, since

$$(q_1(t) \;\; q_2(t))^T(p_1(t) \;\; p_2(t)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we have the orthogonality conditions

$$q_i(t)^T p_j(t) = \delta_{ij} \quad , \tag{2.7}$$

where δ_{ij} denotes the Kronecker's delta symbol. Next, we choose the eigenvectors of A^T to be orthonormal to ζ_i:

$$A^T\zeta_j^* = (-1)^{j-1}\alpha_j\zeta_j^*, \qquad j = 1, 2 \quad \text{and} \quad (\zeta_i^*)^T\zeta_j = \delta_{ij}. \tag{2.8}$$

By combining (2.4), (2.5), (2.7) and (2.8) the solutions of the adjoint problem are seen to satisfy

$$\begin{aligned} &\lim\nolimits_{t\to-\infty} q_1(t)e^{\alpha_1 t} = \zeta_1^*, \qquad \lim\nolimits_{t\to\infty} q_1(t)e^{-\alpha_2 t} = \zeta_2^*, \\ &\lim\nolimits_{t\to-\infty} q_2(t)e^{-\alpha_2 t} = \zeta_2^*, \qquad \lim\nolimits_{t\to\infty} q_2(t)e^{\alpha_1 t} = (1/c)\zeta_1^*, \end{aligned} \tag{2.9}$$

Furthermore, (2.4), (2.7), and (2.9) imply

$$\lim_{t\to\infty} q_2(-t)^T p_1(t)e^{2\alpha_2 t} = 1$$

so that the condition (1.11) is satisfied.

Having established these aymptotic behaviors, we now use the relations

$$\Phi_0(t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Phi_0(s)^{-1} = (p_1(t) \;\; 0)\begin{pmatrix} q_1(s)^T \\ 0 \end{pmatrix},$$

$$\Phi_0(t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Phi_0(s)^{-1} = (0 \;\; p_2(t))\begin{pmatrix} 0 \\ q_2(s)^T \end{pmatrix},$$

to identify exponential dichotomies for (1.5). Defining the projection

$$P_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.10}$$

(2.4), (2.5) and (2.9) imply there is a positive constant K such that

$$\text{(2.11)} \qquad \begin{cases} |\Phi_0(t)P_0\Phi_0(s)^{-1}| \le Ke^{-\alpha_2(t-s)}, & 0 \le s \le t \\ |\Phi_0(t)(I-P_0)\Phi_0(s)^{-1}| \le Ke^{-\alpha_1(s-t)}, & 0 \le t \le s \end{cases}$$

and

$$\text{(2.12)} \qquad \begin{cases} |\Phi_0(t)(I-P_0)\Phi_0(s)^{-1}| \le Ke^{-\alpha_2(t-s)}, & s \le t \le 0 \\ |\Phi_0(t)P_0\Phi_0(s)^{-1}| \le Ke^{-\alpha_1(s-t)}, & t \le s \le 0. \end{cases}$$

Next, we return to the issue of the persistence of bounded solutions near x_h. By substituting $x = x_h(t) + y$ into (1.1), we obtain the following transformed equation for y:

$$\text{(2.13)} \qquad \begin{aligned} &\dot{y} = A(t)y + g(t, y, \mu), \\ &g(t, y, \mu) := f(x_h(t) + y, \mu) - f(x_h(t), 0) - A(t)y. \end{aligned}$$

Noting that $|g| = O(\mu, |y|^2)$, it follows from the above exponential dichotomies and the implicit function theorem that (2.13) has a bounded solution $y = y(t;\mu)$ on $\mathbb{R}$ near $(y, \mu) = (0, 0)$ if and only if

$$M(\mu) := \int_{-\infty}^{\infty} q_2(t)^T g(t, y(t;\mu), \mu)dt = 0.$$

Moreover, we can easily check $y(t;\mu) = O(\mu)$. Hence

$$\text{(2.14)} \qquad M(\mu) = \mu \int_{-\infty}^{\infty} q_2(t)^T \partial_\mu f(x_h(t), 0)dt + O(\mu^2).$$

Under the assumption (1.12),

$$\frac{dM}{d\mu}(0) \ne 0.$$

If $\frac{dM}{d\mu}(0) < 0$ and $\alpha_2 > \alpha_1$, the homoclinic orbit is stable from the inside and stable periodic orbits bifurcate from the homoclinic orbit in the direction $\mu > 0$ (see Fig. 2 and refer to [3], Theorem 10.3.2, p.364, Theorem 11.3.3, p390 and Corollary 11.3.7, p.392 for this bifurcation).

3. Bifurcation of equilibria. In this section, we examine, in detail, the pitchfork bifurcation of the equilibria $(0, 0)$ to which $(x_h(t), x_h(t))$ is homoclinic. Because of the symmetry in the coupling in (1.3), the analysis can be greatly simplified by introducing the transformations

$$\text{(3.1)} \qquad u = \frac{x^1 + x^2}{2}, \qquad v = \frac{x^1 - x^2}{2} \quad .$$

In these new variables, (1.3) is transformed into

$$\text{(3.2)} \qquad \begin{aligned} &\dot{u} = F_1(u, v, \mu) := \frac{1}{2}\{f(u+v, \mu) + f(u-v, \mu)\} \\ &\dot{v} = F_2(u, v, \mu, \nu) := \frac{1}{2}\{f(u+v, \mu) - f(u-v, \mu)\} - 2\nu Dv \end{aligned}$$

Furthermore, we note that $(u, v) = (x_h(t), 0)$ is homoclinic to $(0, 0)$ and corresponds to the symmetric homoclinic solution of (1.3).

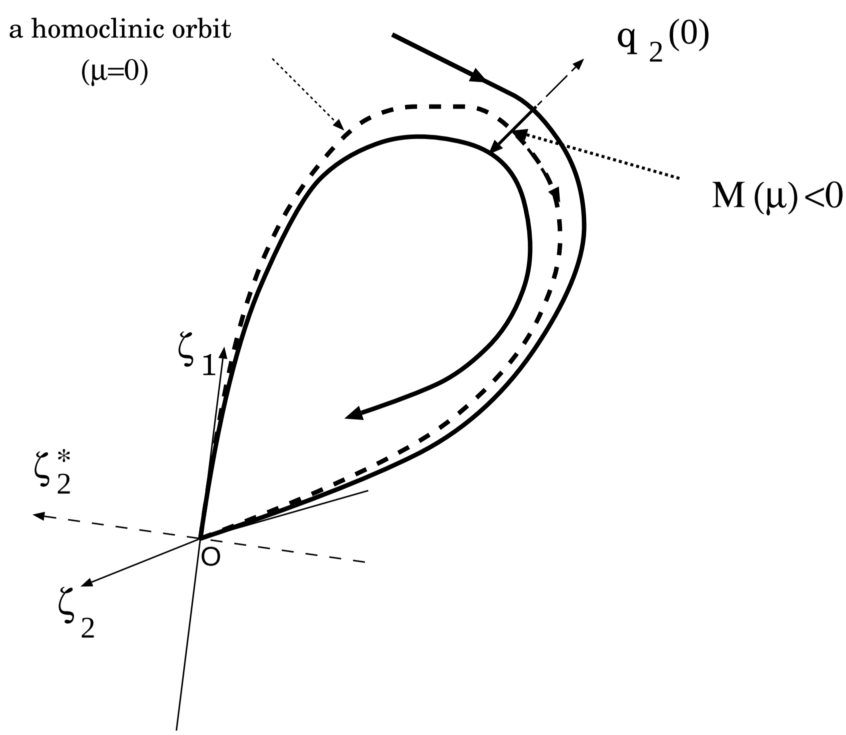

FIG. 2. *The dynamics near the homoclinic orbit*

Next, we summarize a few useful symmetries and derivative expressions of F_j $(j = 1, 2)$ which we use throughout our analyses:

$$\begin{cases} F_1(u,0,\mu) = f(u,\mu), & F_1(u,-v,\mu) = F_1(u,v,\mu) \\ F_2(u,0,\mu,\nu) = 0, & F_2(u,-v,\mu,\nu) = -F_2(u,v,\mu,\nu) \\ \partial_u F_1(\cdot,0,\mu) = \partial_x f(\cdot,\mu), & \partial_v F_1(\cdot,0,\mu) = 0 \\ \partial_u F_2(\cdot,0,\mu) = 0, & \partial_v F_2(\cdot,0,\mu,\nu) = \partial_x f(\cdot,\mu) - 2\nu D. \end{cases} \tag{3.3}$$

Using these expressions, it is easy to verify that the linearized problem around the equilibrium $(u, v) = (0, 0)$ of (3.2) is:

$$\begin{aligned} \dot u &= \partial_x f(0,\mu)u, \\ \dot v &= (\partial_x f(0,\mu) - 2\nu D)v. \end{aligned} \tag{3.4}$$

Note that by introducing the transformations (3.1) the linearized problem (3.4) is now decoupled.

Recalling the assumptions on f outlined in the introduction, the first equation of (3.4) has solutions

$$u(t) = \zeta_1 e^{\alpha_1 t}, \quad \zeta_2 e^{-\alpha_2 t}, \tag{3.5}$$

when $\mu = 0$. Moreover, we assumed that there is a $\nu_h > 0$ at which there is a bifurcation of $(0, 0)$. Specifically, we assume $A - 2\nu_h D$ has the eigenvalues $\alpha_0 = 0$ and $-\alpha_3 < 0$. Letting ζ_0 and ζ_3 be the corresponding eigenvectors:

$$(A - 2\nu_h D)\zeta_0 = 0, \qquad (A - 2\nu_h D)\zeta_3 = -\alpha_3\zeta_3 \quad , \tag{3.6}$$

there are vectors ζ_0^* and ζ_3^* such that

$$(A - 2\nu_h D)^T\zeta_0^* = 0, \qquad (A - 2\nu_h D)^T\zeta_3^* = -\alpha_3\zeta_3^* \tag{3.7}$$

and

$$\zeta_i^{*T}\zeta_j = \delta_{ij}, \qquad i, j = 0, 3. \tag{3.8}$$

Under the assumption (1.8),the implicit function theorem can be used to verify that there are functions $\nu = \nu_0(\mu)$ and $\zeta = \zeta(\mu)$, defined for $|\mu| \ll 1$, satisfying

$$\begin{aligned} &\nu_0(0) = \nu_h, \qquad \zeta(0) = \zeta_0, \\ &(\partial_x f(0,\mu) - 2\nu_0(\mu)D)\zeta(\mu) = 0 \quad . \end{aligned} \tag{3.9}$$

Moreover, for the eigenvalue $\lambda(\mu, \eta)$ of the matrix

$$\partial_x f(0,\mu) - 2(\nu_0(\mu) + \eta)D$$

having $\lambda(\mu, 0) = 0$, one sees that (1.8) implies

$$\frac{\partial\lambda}{\partial\eta}(0,0) = -2\zeta_0^{*T}D\zeta_0 < 0 \quad . \tag{3.10}$$

The above facts imply that on the curve $\nu = \nu_0(\mu)$ in a neighborhood of $(\mu, \nu) = (0, \nu_h)$ the matrix $(\partial_x f(0, \mu) - 2\nu_0(\mu)D)$ has a simple zero eigenvalue and that along curves which cross $\nu = \nu_0(\mu)$ transversely in a direction of decreasing η (or ν), $\lambda(\mu, \eta)$ changes the sign from negative to positive.

Lastly, we look for bifurcating equilibria from $(u, v) = (0, 0)$. Applying standard bifurcation theory (see [10],[18]) to the system of equations

$$F_1(u,v,\mu) = 0, \qquad F_2(u,v,\mu,\nu_0(\mu)+\eta) = 0, \tag{3.11}$$

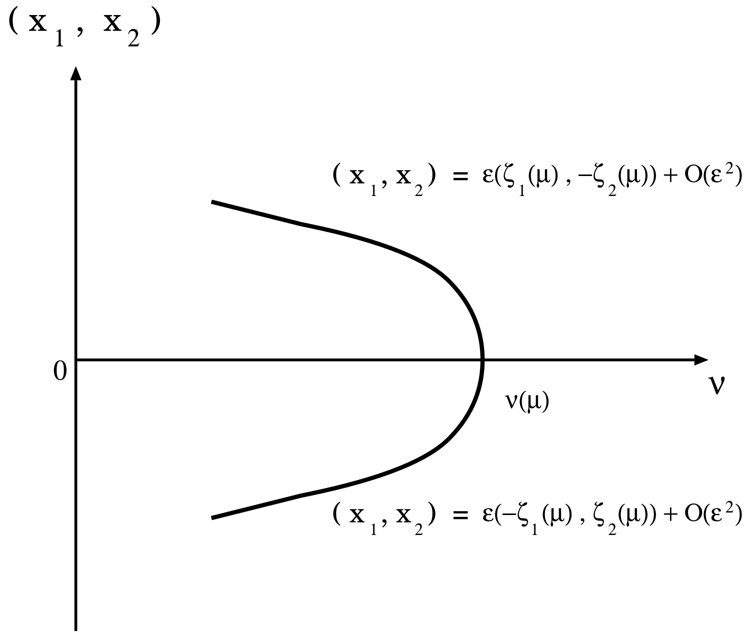

FIG. 3. *Bifurcating equilibria in the* (x_1, x_2) *projection*

we obtain a pair of symmetric bifurcating equilibria

$$(u, v) = (\overline{u}(\mu, \epsilon), \overline{v}(\mu, \epsilon)), \quad (\overline{u}(\mu, \epsilon), -\overline{v}(\mu, \epsilon)) \tag{3.12}$$

for $\eta = \eta(\mu, \epsilon)$, where ϵ is a small parameter. Using the symmetry of these equilibria, it is readily checked that

$$\eta(\mu, -\epsilon) = \eta(\mu, \epsilon), \quad \overline{u}(\mu, -\epsilon) = \overline{u}(\mu, \epsilon), \quad \overline{v}(\mu, -\epsilon) = -\overline{v}(\mu, \epsilon) \tag{3.13}$$

and

$$\eta = O(\epsilon^2), \quad \overline{u}(\cdot, \epsilon) = O(\epsilon^2), \quad \overline{v}(\mu, \epsilon) = \epsilon\zeta(\mu) + O(\epsilon^3). \tag{3.14}$$

It follows from (3.13) and the assumption $(d^2\hat{\nu}/d\epsilon^2)(0) < 0$ of Theorem 1.1 that for each fixed $\mu, |\mu| \ll 1$

$$\frac{\partial \eta}{\partial \epsilon}(\mu, 0) = 0, \qquad \frac{\partial^2 \eta}{\partial \epsilon^2}(\mu, 0) < 0 \quad . \tag{3.15}$$

These bifurcating equilibria are represented schematically in Figure 3. Note that a pitchfork bifurcation of the equilibria occurs in the direction of decreasing η (or ν) for each fixed $\mu, |\mu| << 1$.

4. Exponential dichotomies. Next we consider the linearization of (1.3) about the symmetric homoclinic solution $(x_h(t), x_h(t))$. Using the transformations (3.1) this linearized system is:

$$\dot{\boldsymbol{u}} = \begin{pmatrix} A(t) & 0 \\ 0 & A(t) - 2\nu_h D \end{pmatrix} \boldsymbol{u} \tag{4.1}$$

Because of the block diagonal structure of this system the fundamental solution matrix for (4.1) can be written

$$\Phi(t) := \begin{pmatrix} \Phi_0(t) & 0 \\ 0 & \Phi_1(t) \end{pmatrix} \tag{4.2}$$

where $\Phi_0(t)$ and $\Phi_1(t)$ are fundamental matrices for (1.5) and the linear equation

$$\dot{y} = (A(t) - 2\nu_h D)y \quad , \tag{4.3}$$

respectively. Since the exponential dichotomies of $\dot{p} = A(t)p$ were already identified in section 2, it suffices that we investigate the asymptotic behavior of solutions to (4.3) as $t \to \pm\infty$.

To begin with, we note that since

$$|A(t) - A| = \begin{cases} O(e^{\alpha_1 t}) & \text{as } t \to -\infty \\ O(e^{-\alpha_2 t}) & \text{as } t \to \infty \end{cases} \tag{4.4}$$

Equation (4.3) has a solution $y = p_0(t)$ satisfying

$$\lim_{t\to-\infty} p_0(t) = \zeta_0, \qquad \lim_{t\to\infty} p_0(t) = \hat{c}\zeta_0 \quad , \tag{4.5}$$

for which there are three cases:

$$(i)\ \ \hat{c} > 0, \quad (ii)\ \ \hat{c} < 0, \quad (iii)\ \ \hat{c} = 0 \quad . \tag{4.6}$$

Next we let the fundamental solution matrix to (4.3) be

$$\Phi_1(t) = (p_0(t) \ \ p_3(t)) \quad , \tag{4.7}$$

and take $p_3(t)$ with the asymptotics

$$\lim_{t\to-\infty} p_3(t)e^{\alpha_3 t} = \zeta_3. \tag{4.8}$$

The adjoint equation corresponding to (4.3),

$$\dot{z} + (A(t) - 2\nu_h D)^T z = 0 \tag{4.9}$$

then has a fundamental solution matrix given by

$$(\Phi_1(t)^{-1})^T = (q_0(t) \ \ q_3(t)) \tag{4.10}$$

with

$$q_i(t)^T p_j(t) = \delta_{ij}, \quad i, j = 0, 3. \tag{4.11}$$

It now follows from (4.5), (4.8) and (4.11) that

$$\lim_{t\to-\infty} q_0(t) = \zeta_0^*, \qquad \lim_{t\to-\infty} q_3(t)e^{-\alpha_3 t} = \zeta_3^*. \tag{4.12}$$

The above observation leads us to the following growth estimate: for any $\delta > 0$, there is a $K > 0$ such that

$$|\Phi_1(t)\Phi_1(s)^{-1}| \le Ke^{\delta(t-s)}, \qquad s \le t \tag{4.13}$$

Hence, by defining

$$\begin{aligned} &X(t,s) := \Phi(t)\Phi(s)^{-1}, \\ &P_{cs}(s) = \Phi(s)\begin{pmatrix} P_0 & 0 \\ 0 & I \end{pmatrix}\Phi(s)^{-1} \\ &Q_u(s) = \Phi(s)\begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\Phi(s)^{-1} \end{aligned} \tag{4.14}$$

where P_0 is defined in (2.10), we get to the the following dichotomies for (4.1):

$$\begin{cases} |X(t,s)P_{cs}(s)| \le Ke^{\delta(t-s)}, & 0 \le s \le t \\ |X(t,s)(I - P_{cs}(s))| \le Ke^{-\alpha_1(s-t)}, & 0 \le t \le s \end{cases} \tag{4.15}$$

(4.16)
$$\begin{cases} |X(t,s)(I-Q_u(s))| \le Ke^{\delta(t-s)}, & s \le t \le 0 \\ \\ |X(t,s)Q_u(s))| \le Ke^{-\alpha_1(s-t)}, & t \le s \le 0 \end{cases}$$

(recall (2.11) and (2.12)).

Finally we remark that (4.5), (4.8) and (4.11) imply that no solution to the adjoint equation (4.9) has exponential decay as $t \to \infty$. More precisely, for any positive $\gamma < \alpha_3$ every solution $q(t)$ of (4.9) satisfies

(4.17)
$$\lim_{t\to\infty} q(t)e^{\gamma t} = \infty.$$

This fact will be also used for deriving the bifurcation equation in the next section.

5. Fredholm alternative for the homoclinic bifurcation. In this section the Liapunov-Schmidt reduction is carried out and the bifurcation equation is identified. The asymptotic behavior of the resulting solutions is also analyzed. The analysis of the bifurcation equation is given in the next section.

For the analysis of the bifurcating equilibria $(\bar{u}, \bar{v})$ in §3, it became convenient to parametrize the bifurcating curve in (μ, ε) (as opposed to (μ, ν)). We continue to use this parametrization here and define

(5.1)
$$\overline{\boldsymbol{u}}(\lambda) := \begin{pmatrix} \overline{u}(\lambda) \\ \overline{v}(\lambda) \end{pmatrix}, \qquad \lambda := (\mu, \epsilon) \quad .$$

Recalling that the bifurcation of equilibria occurs on the curve $\nu = \nu_0(\mu)$, the system (3.2) at these points can be written as

(5.2)
$$\dot{\boldsymbol{u}} = F(\boldsymbol{u}, \lambda) := \begin{pmatrix} F_1(u,v,\mu) \\ F_2(u,v,\mu,\nu_0(\mu)+\eta(\mu,\epsilon)) \end{pmatrix}.$$

Note

(5.3)
$$F(0,\lambda) = 0, \qquad F(\overline{\boldsymbol{u}}(\lambda), \lambda) = 0, \quad for\ |\lambda| \ll 1.$$

Moreover, the symmetry in the system (5.2) can be characterized by

(5.4)
$$R_1 F(\boldsymbol{u}, \lambda) = F(R_1\boldsymbol{u}, \lambda) = F(R_1\boldsymbol{u}, r_1(\lambda))$$

where

(5.5)
$$R_1 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad r_1(\lambda) := (\mu, -\epsilon).$$

Now we look for a bounded solution $\boldsymbol{u}(t)$ of (5.2) satisfying

(5.6)
$$\lim_{t\to-\infty} \boldsymbol{u}(t) = \overline{\boldsymbol{u}}(\lambda)$$

near the symmetric homoclinic solution $(u, v) = (x_h(t), 0)$. First, set

(5.7)
$$\boldsymbol{u} = \boldsymbol{u}_h(t) + \overline{\boldsymbol{u}}(\lambda) + \boldsymbol{z}, \qquad \boldsymbol{u}_h(t) := \begin{pmatrix} x_h(t) \\ 0 \end{pmatrix}.$$

Substituting (5.7) into (5.2) yields

(5.8)
$$\dot{\boldsymbol{z}} = \boldsymbol{A}(t)\boldsymbol{z} + \boldsymbol{G}(\boldsymbol{z}, \lambda)(t) \quad ,$$

where

(5.9)
$$\boldsymbol{A}(t) := \frac{\partial}{\partial \boldsymbol{u}} F(\boldsymbol{u}_h(t), 0) = \begin{pmatrix} A(t) & 0 \\ 0 & A(t) - 2\nu_h D \end{pmatrix}$$
$$\boldsymbol{G}(\boldsymbol{z}, \lambda) := F(\boldsymbol{u}_h(t) + \overline{\boldsymbol{u}}(\lambda) + \boldsymbol{z}, \lambda) - F(\boldsymbol{u}_h(t), 0) - \boldsymbol{A}(t)\boldsymbol{z}.$$

We note that

$$F(\boldsymbol{u}_h(t),0) = \begin{pmatrix} f(x_h(t),0) \\ 0 \end{pmatrix},$$

and

$$\boldsymbol{G}(0,0)(t) \equiv 0, \qquad |\boldsymbol{G}(0,\lambda)(t)| = \begin{cases} O(e^{\alpha_1 t}), & t \to -\infty \\ O(e^{-\alpha_2 t}), & t \to \infty \end{cases} \tag{5.10}$$

and that by (3.13) and (5.4)

$$R_1 \boldsymbol{G}(\boldsymbol{z},\lambda) = \boldsymbol{G}(R_1 \boldsymbol{z}, r_1(\lambda)) \quad . \tag{5.11}$$

Let $\chi(s)$ ($s \in \mathbb{R}$) be a C^∞ function satisfying

$$\chi(s) = \begin{cases} 1, & s \le 1, \\ 0, & s \ge 4. \end{cases}$$

Given $\rho > 0$, we modify the function $\boldsymbol{G}(\boldsymbol{z},\lambda)$ as follows:

$$\boldsymbol{G}_\rho(\boldsymbol{z},\lambda) = \chi(|\boldsymbol{z}|^2/\rho^2)\boldsymbol{G}(\boldsymbol{z},\lambda). \tag{5.12}$$

Then the symmetry property (5.11) is preserved for $\boldsymbol{G}_\rho(\boldsymbol{z},\lambda)$:

$$R_1 \boldsymbol{G}_\rho(\boldsymbol{z},\lambda) = \boldsymbol{G}_\rho(R_1 \boldsymbol{z}, r_1(\lambda)). \tag{5.13}$$

Moreover,

$$\boldsymbol{G}_\rho(0,0)(t) \equiv 0, \qquad |\boldsymbol{G}_\rho(0,\lambda)(t)| = \begin{cases} O(e^{\alpha_1 t}), & t \to -\infty \\ O(e^{-\alpha_2 t}), & t \to \infty \end{cases} \tag{5.14}$$

hold.

For γ satisfying $\delta < \gamma < \alpha_j$ ($j = 1,2,3$), define

$$\begin{aligned} C_\gamma &:= \{\boldsymbol{u} \in C(\mathbb{R};\mathbb{R}^4) : \|\boldsymbol{u}\|_\gamma := \sup\nolimits_{-\infty<t<\infty} e^{-\gamma t}|\boldsymbol{u}(t)| < \infty\} \\ C^1_\gamma &:= \{\boldsymbol{u} \in C^1(\mathbb{R};\mathbb{R}^4) : \boldsymbol{u} \in C_\gamma \text{ and } d\boldsymbol{u}/dt \in C_\gamma\} \end{aligned} \tag{5.15}$$

Both C_γ and C^1_γ are Banach spaces with respective norms $\|\boldsymbol{u}\|_\gamma$ and $\|\boldsymbol{u}\|_{\gamma,1} := \|\boldsymbol{u}\|_\gamma + \|d\boldsymbol{u}/dt\|_\gamma$. Using (5.14) and the boundedness of the derivative of $\boldsymbol{G}_\rho$ it is evident that $\boldsymbol{G}_\rho(\boldsymbol{z},\lambda)(\cdot) \in C_\gamma$ for $\boldsymbol{z} \in C_\gamma$. Furthermore, $\boldsymbol{G}_\rho$ is Lipschitz continuous since

$$\|\boldsymbol{G}_\rho(\boldsymbol{z}_1,\lambda) - \boldsymbol{G}_\rho(\boldsymbol{z}_2,\lambda)\|_\gamma \le K_1(\rho,\lambda)\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_\gamma \text{ for } \boldsymbol{z}_i \in C_\gamma \ (i = 1,2), \tag{5.16}$$

where

$$K_1(\rho,\lambda) := \sup_{\boldsymbol{z}\in\mathbb{R}^4} \left| \frac{\partial \boldsymbol{G}_\rho(\boldsymbol{z})}{\partial \boldsymbol{z}} \right|.$$

In addition we can verify

$$K_1(\rho,\lambda) \to 0, \text{ as } (\rho,\lambda) \to (0,0). \tag{5.17}$$

Next we define the linear operator

$$\mathcal{F}[\boldsymbol{z}] := \dot{\boldsymbol{z}} - \boldsymbol{A}(t)\boldsymbol{z}, \tag{5.18}$$

where $\mathcal{F} : \mathcal{D}(\mathcal{F}) \to C_\gamma$ and the domain $\mathcal{D}(\mathcal{F}) = C^1_\gamma$. Lemma 3.2 of [4] implies that $\mathcal{F}$ is a Fredholm operator with $\mathrm{Ind}(\mathcal{F}) = 0$ and has the range

$$\mathcal{R}(\mathcal{F}) = \left\{ \boldsymbol{h} \in C_\gamma : \int_{-\infty}^{\infty} \psi(t)^T \boldsymbol{h}(t)dt = 0, \forall \psi \in C_{-\gamma} \ s.t. \ \dot{\psi} + \boldsymbol{A}(t)^T\psi = 0 \right\} \tag{5.19}$$

(see also Lemma 4.5 of [9]). Using (2.9) and (4.17), it is easily verified that

$$\psi(t) := \begin{pmatrix} q_2(t) \\ 0 \end{pmatrix} \tag{5.20}$$

is the unique solution in $C_{-\gamma}$ (up to constant multiplication) of the homogeneous adjoint equation

$$\dot{\boldsymbol{v}} + \boldsymbol{A}(t)^T \boldsymbol{v} = \begin{pmatrix} \dot{v}^1 + A(t)^T v^1 \\ \dot{v}^2 + (A(t) - 2\nu_h D)^T v^2 \end{pmatrix} = 0 \quad . \tag{5.21}$$

Therefore,

$$\mathcal{F}[\boldsymbol{z}] = \boldsymbol{h} := \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} \in C_\gamma \tag{5.22}$$

has a (nontrivial) solution if and only if

$$\int_{-\infty}^{\infty} \psi(t)^T \boldsymbol{h}(t) dt = \int_{-\infty}^{\infty} q_2(t)^T h^1(t) dt = 0. \tag{5.23}$$

Now let $\mathcal{Q}$ be a projection onto $\mathcal{R}(\mathcal{F})$ and let $\mathcal{P}$ be a projection onto the nullspace of $\mathcal{F}$:

$$\mathcal{N}(\mathcal{F}) = \{\boldsymbol{u} : \boldsymbol{u} = c\dot{\boldsymbol{u}}_h, \quad c \in \mathbb{R}\} \quad .$$

We note that $\mathcal{Q}C_\gamma$ is closed and the restriction of $\mathcal{F}$ to $(I - \mathcal{P})C^1_\gamma$ is 1-1 onto $\mathcal{Q}C_\gamma$. It thereby follows that the restricted operator has a continuous inverse which we denote by $\mathcal{F}^{-1}$.

Having established these facts and definitions, the solution of the mollified equation,

$$\mathcal{F}[\boldsymbol{z}] = \boldsymbol{G}_\rho(\boldsymbol{z}, \lambda) \tag{5.24}$$

can be decomposed as follows:

$$\begin{aligned} &\boldsymbol{z} = a\dot{\boldsymbol{u}}_h + \hat{\boldsymbol{z}}, \quad \hat{\boldsymbol{z}} \in (I - \mathcal{P})C^1_\gamma, \\ &\hat{\boldsymbol{z}} = \mathcal{F}^{-1}\mathcal{Q}\boldsymbol{G}_\rho(a\dot{\boldsymbol{u}}_h + \hat{\boldsymbol{z}}, \lambda), \\ &\int_{-\infty}^{\infty} \psi(t)^T \boldsymbol{G}_\rho(a\dot{\boldsymbol{u}}_h + \hat{\boldsymbol{z}}, \lambda)(t) dt = 0. \end{aligned} \tag{5.25}$$

Since the original system is autonomous, we can fix the phase of the solution sought for by assuming that $a = 0$ in (5.25). Thus, we seek a solution $\hat{\boldsymbol{z}}$ of

$$\hat{\boldsymbol{z}} = \mathcal{F}^{-1}\mathcal{Q}\boldsymbol{G}_\rho(\hat{\boldsymbol{z}}, \lambda), \tag{5.26}$$

$$\int_{-\infty}^{\infty} \psi(t)^T \boldsymbol{G}_\rho(\hat{\boldsymbol{z}}, \lambda)(t) dt = 0. \tag{5.27}$$

Given (5.17), standard fixed point theory implies that for $\rho > 0$ sufficiently small, there is a unique solution $\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}(\lambda)$ of (5.26) in a small neighborhood of $(\hat{\boldsymbol{z}}, \lambda) = (0, 0)$. Considering (5.10) and (5.11), such a solution $\hat{\boldsymbol{z}}(\lambda)$ must satisfy

$$\hat{\boldsymbol{z}}(0) = 0, \qquad R_1\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}(r_1(\lambda)) \quad . \tag{5.28}$$

Substituting $\hat{\boldsymbol{z}}(\lambda)$ into (5.27) yields the bifurcation equation

$$H(\lambda) := \int_{-\infty}^{\infty} \psi(t)^T \boldsymbol{G}_\rho(\hat{\boldsymbol{z}}(\lambda), \lambda)(t) dt = 0 \quad . \tag{5.29}$$

From (5.28) and (5.20), it is easily verified that

$$H(0)=0, \qquad H(r_1(\lambda))=H(\lambda) \quad . \tag{5.30}$$

In section 6, we prove that there exist an $\epsilon_0>0$ and a function $\mu=\mu(\epsilon)$ such that

$$\mu(0)=0, \qquad H(\mu(\epsilon),\epsilon)=0 \quad \text{for each} \quad \epsilon\in(0,\epsilon_0).$$

For this $\mu(\epsilon)$, $\hat{\boldsymbol{z}}(\mu(\epsilon),\epsilon)$ is a solution to (5.24) with the asymptotics

$$\|\hat{\boldsymbol{z}}(\mu(\epsilon),\epsilon)\|_\gamma = O(\epsilon) \quad (\epsilon\to 0). \tag{5.31}$$

Furthermore,

$$\boldsymbol{u}_\epsilon(t) := \boldsymbol{u}_h(t)+\overline{\boldsymbol{u}}(\mu(\epsilon),\epsilon)+[\hat{\boldsymbol{z}}(\mu(\epsilon),\epsilon)](t)$$

is a solution of (5.2) if

$$\sup_{t\in\mathbb{R}}|[\hat{\boldsymbol{z}}(\mu(\epsilon),\epsilon)](t)|<\rho, \tag{5.32}$$

Also note that $\boldsymbol{u}_{-\epsilon}$ is also a solution which is symmetric to $\boldsymbol{u}_\epsilon(t)$.

To show (5.32) is satisfied and $\boldsymbol{u}_\epsilon(t)$ is the desired solution of (5.2), we need first discuss the behavior of $\boldsymbol{u}_\epsilon(t)$ as $t\to\infty$. Given (5.31), there is a positive constant $\kappa_0>0$ such that

$$|\hat{\boldsymbol{z}}(\mu(\epsilon),\epsilon)(t)| \le \kappa_0\epsilon e^{\gamma t} \quad (-\infty<t<\infty).$$

There also exists a number $\kappa_1>0$ for which

$$|\boldsymbol{u}_h(t)| \le \kappa_1 e^{-\alpha_2 t} \quad (t>0).$$

Define $T_0=T_0(\epsilon)$ as

$$T_0=\frac{1}{\alpha_2}\log(\kappa_1/\epsilon).$$

Then

$$|[\hat{\boldsymbol{z}}(\mu(\epsilon),\epsilon)](t)| \le \kappa_0\epsilon(\kappa_1/\epsilon)^{\gamma/\alpha_2}=\kappa_0\kappa_1^{\gamma/\alpha_2}\epsilon^{1-\gamma/\alpha_2}, \quad -\infty<t\le T_0,$$

from which we deduce $|[\hat{\boldsymbol{z}}(\mu(\epsilon),\epsilon)](t)|<\rho,\ t\in(-\infty,T_0]$ for a sufficiently small $\epsilon>0$ (recalling $\alpha_2>\gamma$). Furthermore

$$|\boldsymbol{u}_\epsilon(T_0)| \le \epsilon+|\overline{\boldsymbol{u}}(\mu(\epsilon),\epsilon)|+\kappa_0\kappa_1^{\gamma/\alpha_2}\epsilon^{1-\gamma/\alpha_2}=O(\epsilon)+O(\epsilon^{1-\gamma/\alpha_2}). \tag{5.33}$$

Applying invariant manifold theory (for instance see [19] and [6]), there is a local center-stable manifold $W^{cs}(0)$ for (5.2) in a neighborhood $\mathcal{U}(0)$, which depends smoothly on ϵ. Furthermore $W^{cs}(0)$ contains a center manifold which is foliated by stable leaves $\{V(\boldsymbol{v}) : \boldsymbol{v}\in W^c(0)\}$. More precisely, for any solution initialized in a leaf, say $\boldsymbol{u}(t;\boldsymbol{u}_0), \boldsymbol{u}_0\in V(\boldsymbol{v}_0)$, the distance between the two solutions $|\boldsymbol{u}(t;\boldsymbol{u}_0)-\boldsymbol{u}(t;\boldsymbol{v}_0)|$ decays exponentially in $\mathcal{U}(0)$, where $\boldsymbol{u}(t;\boldsymbol{v}_0)$ is a solution on $W^c(0)$ (see [5]). Indeed, there are constants $\beta_1>0$ and $M_1>0$ such that

$$|\boldsymbol{u}(t;\boldsymbol{u}_0)-\boldsymbol{u}(t;\boldsymbol{v}_0)| \le M_1e^{-\beta_1 t}|\boldsymbol{u}_0-\boldsymbol{v}_0| \qquad (\boldsymbol{u}_0\in V(\boldsymbol{v}_0))$$

holds for all $t\ge 0$ (as long as the solutions remain in $\mathcal{U}(0)$). Furthermore, the equilibria $\boldsymbol{u}=0$, $\overline{\boldsymbol{u}}(\mu(\epsilon),\epsilon)$ and $R_1\overline{\boldsymbol{u}}(\mu(\epsilon),\epsilon)$ must lie in the center manifold. On $W^{cs}(0)$ the first of these equilibria is unstable while the remaining ones are asymptotically stable. Then, if $\boldsymbol{u}_\epsilon(T_0)\in W_{cs}(0)$, we can estimate

$$\begin{aligned}|\boldsymbol{u}_\epsilon(t+T_0)| &\le |\boldsymbol{u}_\epsilon(t+T_0)-\boldsymbol{u}(t;\boldsymbol{v}_0)|+|\boldsymbol{u}(t;\boldsymbol{v}_0)| \\ &\le M_1e^{-\beta_1 t}|\boldsymbol{u}_\epsilon(T_0)-\boldsymbol{v}_0|+|\boldsymbol{u}(t;\boldsymbol{v}_0)| \quad (\boldsymbol{u}_\epsilon(T_0)\in V(\boldsymbol{v}_0), \boldsymbol{v}_0\in W^c(0)).\end{aligned}$$

Since each leaf is expressed by a graph of a smooth function, whose derivative has a uniform bound in ϵ, we see that

$$\boldsymbol{v}_0 = O(\boldsymbol{u}_\epsilon(T_0)) = O(\epsilon^{1-\gamma/\alpha_2}).$$

We thereby conclude that (5.32) is satisfied and that $\boldsymbol{u}_\epsilon(t)$ converges to one of the three equilibria provided $\boldsymbol{u}_\epsilon(T_0) \in W^{cs}(0)$ for each small $\epsilon > 0$.

To see that $\boldsymbol{u}_\epsilon(T_0) \in W^{cs}(0)$ for small $\epsilon > 0$. we assume that $\boldsymbol{u}_\epsilon(T_0) \notin W^{cs}(0)$ to obtain a contradiction. Then $\boldsymbol{u}_\epsilon(t)$ must eventually leave the ball $\{\boldsymbol{u} : |\boldsymbol{u}| < 3\rho/4\}$. Otherwise it must belong to to $W^{cs}(0)$ for $t \geq T_0$ since $W^{cs}(0)$ contains any bounded solution for $t \geq T_0$. We let $T_1 = T_1(\epsilon) > T_0$ be a number realizing $|\boldsymbol{u}_\epsilon(T_1)| = 3\rho/4$. Recall that the $W^{cs}(0)$ is expressed by the graph of a function $\varphi : \tilde{P}_{cs}\mathbb{R}^4 \to (I - \tilde{P}_{cs})\mathbb{R}^4$ in $\mathcal{U}(0)$, where $\tilde{P}_{cs}$ is the projection

$$\tilde{P}_{cs} := \begin{pmatrix} (\zeta_2 \;\; 0)\begin{pmatrix} (\zeta_2^*)^T \\ 0 \end{pmatrix} & 0 \\ 0 & I \end{pmatrix}.$$

We again apply invariant manifold theory to assert that in the neighborhood $\mathcal{U}(0)$ there are unstable invariant leaves $\{V^u(\boldsymbol{v}) : \boldsymbol{v} \in W^{cs}(0)\}$ which foliate $\mathcal{U}(0)$. This and the previously observed flow on $W^{cs}(0)$ imply

$$|(I - \tilde{P}_{cs})\boldsymbol{u}_\epsilon(T_1)| > \rho/2 \tag{5.34}$$

for small $\epsilon > 0$.

On the other hand, using the variation-of-constants formula and the dichotomy in (4.15), we can easily verify that for $t \geq 0$, $[\hat{\boldsymbol{z}}(\lambda(\epsilon))](t)$, $\lambda(\epsilon) := (\mu(\epsilon), \epsilon)$, is a solution of the following integral equation:

$$\begin{aligned} \boldsymbol{z}(t) = \;& X(t,0)P_{cs}(0)\xi + \int_0^t X(t,s)P_{cs}(s)G_\rho(\boldsymbol{z}, \lambda(\epsilon))(s)ds \\ & + \int_\infty^t X(t,s)(I - P_{cs}(s))G_\rho(\boldsymbol{z}, \lambda(\epsilon))(s)ds, \quad \xi \in \mathbb{R}^4 \end{aligned} \tag{5.35}$$

where $X(t,s)$ and $P_{cs}(s)$ (both defined in (4.14)) are the fundamental solution matrix to (4.1) and a projection, respectively. Since

$$|G_\rho(\boldsymbol{z}, \lambda(\epsilon))| \leq |G_\rho(\boldsymbol{z}, 0)| + |G_\rho(\boldsymbol{z}, \lambda(\epsilon)) - G_\rho(\boldsymbol{z}, 0)| \leq K_1(\rho^2 + \epsilon)$$

for some constant $K_1 > 0$ and

$$(I - P_{cs}(t))[\hat{z}(\lambda(\epsilon))](t) = \int_\infty^t X(t,s)(I - P_{cs}(s))G_\rho(\hat{\boldsymbol{z}}(\lambda(\epsilon)), \lambda(\epsilon))(s)ds,$$

we obtain

$$|(I - P_{cs}(t))[\hat{z}(\lambda(\epsilon))](t)| \leq \frac{KK_1}{\alpha_1}(\rho^2 + \epsilon). \tag{5.36}$$

Moreover, for some constant $M_2 > 0$

$$|P_{cs}(t) - \tilde{P}_{cs}| \leq M_2 e^{-\alpha_2 t}, \quad t > 0. \tag{5.37}$$

Indeed, since

$$P_{cs}(t) = \begin{pmatrix} (p_1(t) \;\; 0)\begin{pmatrix} q_1(t)^T \\ 0 \end{pmatrix} & 0 \\ 0 & I \end{pmatrix},$$

(5.37) holds provided that $|p_1(t)e^{\alpha_2 t} - \zeta_2|, |q_1(t)e^{-\alpha_2 t} - \zeta_2^*| = O(e^{-\alpha_2 t})$ as $t \to \infty$. The first of these asymptotic behaviors follows from the fact that $p_1(t) = \dot{x}_h(t)$ and $x_h(t) = c_2\zeta_2 e^{-\alpha_2 t} + O(e^{-2\alpha_2 t})$.

To demonstrate $|q_1(t)e^{-\alpha_2 t} - \zeta_2^*| = O(e^{-\alpha_2 t})$ as $t \to \infty$, recall that $q_1(t)$ satisties (2.9) and is a solution of

$$\dot{q} = -A(t)^T q = -A^T q + B(t)q, \text{ where } B(t) := A(t)^T - A^T.$$

We can write $q_1(t) = \xi_1(t)\zeta_1^* + \xi_2(t)\zeta_2^*$, where ξ_1 and ξ_2 are given by:

$$\begin{aligned}
\xi_1(t) &= e^{-\alpha_1 t}\xi_1(0) + \int_0^t e^{-\alpha_1(t-s)}\zeta_1^T B(s)q_1(s)ds,\\
\xi_2(t) &= e^{\alpha_2 t} + \int_\infty^t e^{\alpha_2(t-s)}\zeta_2^T B(s)q_1(s)ds = e^{\alpha_2 t}\left(1 + \int_\infty^t \zeta_2^T B(s)q_1(s)e^{\alpha_2 s}ds\right).
\end{aligned}$$

By (4.4), we find that $|B(t)| = O(e^{-\alpha_2 t})$ as $t \to \infty$. It follows from this estimate and (2.9) that $\xi_1(t) = O(1)$ and $|\xi_2(t)e^{-\alpha_2 t} - 1| = O(e^{-\alpha_2 t})$ as $t \to \infty$ from which the asymptotic estimate for $q_1(t)$ is easily obtained.

By (5.36),(5.37) and $T_1 > T_0 = (1/\alpha_2)\log(\kappa_1/\epsilon)$, we obtain

$$\begin{aligned}
|(I - \tilde{P}_{cs})\boldsymbol{u}_\epsilon(T_1)| &= O(\epsilon) + |(I - \tilde{P}_{cs})[\hat{z}(\lambda(\epsilon))](T_1)|\\
&\le O(\epsilon) + \frac{KK_1}{\alpha_1}\rho^2 + |(P_{cs}(T_1) - \tilde{P}_{cs})[\hat{z}(\lambda(\epsilon))](T_1)|\\
&= O(\epsilon) + \frac{KK_1}{\alpha_1}\rho^2 + O(\epsilon^{1-\gamma/\alpha_2}).
\end{aligned}$$

Now choose $\rho > 0$ small enough so that $\rho KK_1/\alpha_1 < 1/4$ holds, and fix it. Then, for sufficiently small $\epsilon > 0$, $|(I - \tilde{P}_{cs})\boldsymbol{u}_\epsilon(T_1)| < \rho/2$ which is direct contradiction to (5.34). Thus, we can assert that $\boldsymbol{u}_\epsilon(t) \in W^{cs}(0), t \ge T_0$.

Now there are three cases:

(i)	$\lim_{t\to\infty} \boldsymbol{u}_\epsilon(t) = \overline{\boldsymbol{u}}(\mu(\epsilon), \epsilon)$:	a homoclinic orbit asymptotic to a bifurcating equilibrium;
(ii)	$\lim_{t\to\infty} \boldsymbol{u}_\epsilon(t) = R_1\overline{\boldsymbol{u}}(\mu(\epsilon), \epsilon)$:	a heteroclinic orbit connecting a bifurcating equilibrium with the other one;
(iii)	$\lim_{t\to\infty} \boldsymbol{u}_\epsilon(t) = 0$:	a heteroclinic orbit connecting a bifurcating equilibrium with the trivial one.

In the next section we show that the solution $\boldsymbol{u}_\epsilon(t)$ can be expanded as

$$\boldsymbol{u}_\epsilon(t) = \boldsymbol{u}_h(t) + \epsilon(0, p_0(t))^T + O(\epsilon^2) \tag{5.38}$$

where $p_0(t)$ is the bounded solution with asymptotics described in (1.10). To see how the sign of $\hat{c}$ distinguishes between cases (i) and (ii) above, recall $\boldsymbol{u}_h(t) \to 0$, $p_0(t) \to \hat{c}\zeta_0$ as $t \to \infty$ and $\overline{\boldsymbol{u}}(\mu(\epsilon), \epsilon) = \epsilon\zeta_0 + O(\epsilon^2)$. These facts show how $\hat{c} > 0$ (resp. < 0) implies the first case (i) (resp. the second one (ii)). Indeed for sufficiently large $T > 0$, the orbit $\boldsymbol{u}_\epsilon(t)$ enters the basin of the stable equilibrium $\overline{\boldsymbol{u}}(\mu(\epsilon), \epsilon)$ (resp. $R_1\overline{\boldsymbol{u}}(\mu(\epsilon), \epsilon)$) relative to $W^{cs}(0)$ provided ϵ is small enough. Lastly we note the leading-order analysis presented in the next section is not sufficient to make a connection between the case $\hat{c} = 0$ and conclusion (iii).

6. The bifurcation equation. In this section, we solve the bifurcation equation (5.29) and prove Theorem 1.1. First note that $\hat{\boldsymbol{z}}(\lambda)$ is C^2 if f (that is F) is C^3 provided γ satisfies $2\delta < 2\gamma < \alpha_1, \alpha_3$ (see [5], [4]). This smoothness is required to apply the implicit function theorem.

Differentiating the first equation of (5.25) in λ, we obtain

$$\partial_\lambda \hat{\boldsymbol{z}}(\lambda) = \mathcal{F}^{-1}\mathcal{Q}\left\{\frac{\partial}{\partial \boldsymbol{z}}\boldsymbol{G}_\rho(\hat{\boldsymbol{z}}(\lambda), \lambda)\partial_\lambda\hat{\boldsymbol{z}}(\lambda) + \frac{\partial}{\partial\lambda}\boldsymbol{G}_\rho(\hat{\boldsymbol{z}}(\lambda), \lambda)\right\}.$$

From the definition of $\boldsymbol{G}_\rho$

$$\frac{\partial}{\partial \boldsymbol{z}}\boldsymbol{G}_\rho(0,0) = 0, \quad \frac{\partial}{\partial\lambda}\boldsymbol{G}_\rho(0,0) = \boldsymbol{A}(t)\partial_\lambda\overline{\boldsymbol{u}}(\lambda) + \partial_\lambda F(\boldsymbol{u}_h(t), 0). \tag{6.1}$$

Since

$$\partial_\mu F(\boldsymbol{u}_h(t),0,0) = \begin{pmatrix} \partial_\mu f(x_h(t),0) \\ 0 \end{pmatrix}, \quad \partial_\epsilon F(\boldsymbol{u}_h(t),0,0) = 0,$$

we obtain

$$\begin{aligned} \partial_\mu \hat{\boldsymbol{z}}(0,0) &= \mathcal{F}^{-1}\mathcal{Q}\left\{ \boldsymbol{A}(t)\partial_\mu \overline{\boldsymbol{u}}(0,0) + \begin{pmatrix} \partial_\mu f(x_h(t),0) \\ 0 \end{pmatrix}\right\} \\ \partial_\epsilon \hat{\boldsymbol{z}}(0,0) &= \mathcal{F}^{-1}\mathcal{Q}(\boldsymbol{A}(t)\partial_\epsilon \overline{\boldsymbol{u}}(0,0)). \end{aligned} \tag{6.2}$$

Differentiating $H(\mu,\epsilon)$ in μ and setting $(\mu,\epsilon)=(0,0)$ we obtain:

$$\begin{aligned} \partial_\mu H(0,0) &= \int_{-\infty}^{\infty} \psi(t)^T \left\{ \frac{\partial}{\partial \boldsymbol{u}} F(\boldsymbol{u}_h(t),0,0)(\partial_\mu \overline{\boldsymbol{u}}(0,0)) + \partial_\mu \hat{\boldsymbol{z}}(0,0)) \right. \\ &\qquad \left. + \partial_\mu F(\boldsymbol{u}_h(t),0,0) - \boldsymbol{A}(t)\partial_\mu \hat{\boldsymbol{z}}(0,0) \right\} dt \\ &= \int_{-\infty}^{\infty} q_2(t)^T\{A(t)\partial_\mu \overline{u}(0,0) + \partial_\mu f(x_h(t),0)\}dt \\ &= \int_{-\infty}^{\infty} \{-\dot{q}_2(t)^T\partial_\mu \overline{u}(0,0) + q_2(t)^T\partial_\mu f(x_h(t),0)\}dt \\ &= \int_{-\infty}^{\infty} q_2(t)^T \partial_\mu f(x_h(t),0)dt = h_1 \neq 0 \end{aligned} \tag{6.3}$$

(recall (1.12)). Thus, by the implicit function theorem there exists a C^2-function $\mu=\mu(\epsilon)$ $|\epsilon| \ll 1$ such that

$$\mu(0)=0, \qquad H(\mu(\epsilon),\epsilon)=0.$$

Considering (5.30), we see $\mu(-\epsilon)=\mu(\epsilon)$. Hence we can write

$$\mu(\epsilon) = \frac{1}{2}\mu_2\epsilon^2 + o(\epsilon^2) \quad . \tag{6.4}$$

By expanding $H(\mu,\epsilon)$ as

$$H(\mu,\epsilon) = \partial_\mu H(0,0)\mu + \frac{1}{2}\partial_\epsilon^2 H(0,0)\epsilon^2 + \text{h.o.t}$$

we obtain

$$\mu_2 = -\partial_\epsilon^2 H(0,0)/\partial_\mu H(0,0). \tag{6.5}$$

In the rest of this section we compute $\partial_\epsilon^2 H(0,0)$.

$$\partial_\epsilon^2 H(0,0) = \int_{-\infty}^{\infty} \psi(t)^T \left\{ \frac{\partial^2}{\partial \boldsymbol{u}^2} F(\boldsymbol{u}_h(t),0,0)) \circ (\partial_\epsilon \overline{\boldsymbol{u}}(0,0) + \partial_\epsilon \hat{\boldsymbol{z}}(0,0), \right.$$

$$\left. \partial_\epsilon \overline{\boldsymbol{u}}(0,0) + \partial_\epsilon \hat{\boldsymbol{z}}(0,0)) + \frac{\partial}{\partial \boldsymbol{u}} F(\boldsymbol{u}_h(t),0,0) \partial_\epsilon^2 \overline{\boldsymbol{u}}(0,0) \right\} dt$$

$$= \int_{-\infty}^{\infty} q_2(t)^T \frac{\partial^2}{\partial \boldsymbol{u}^2} F^1(\boldsymbol{u}_h(t),0) \circ (\partial_\epsilon \overline{\boldsymbol{u}}(0,0) + \partial_\epsilon \hat{\boldsymbol{z}}(0,0),$$

(6.6)

$$\partial_\epsilon \overline{\boldsymbol{u}}(0,0) + \partial_\epsilon \hat{\boldsymbol{z}}(0,0)) dt - \int_{-\infty}^{\infty} \dot{q}(t)_2^T \partial_\epsilon \overline{u}(0,0) dt$$

$$= \int_{-\infty}^{\infty} q_2(t)^T \frac{\partial^2}{\partial \boldsymbol{u}^2} F^1(\boldsymbol{u}_h(t),0) \circ (\partial_\epsilon \overline{\boldsymbol{u}}(0,0) + \partial_\epsilon \hat{\boldsymbol{z}}(0,0),$$

$$\partial_\epsilon \overline{\boldsymbol{u}}(0,0) + \partial_\epsilon \hat{\boldsymbol{z}}(0,0)) dt$$

where we used

$$\psi(t)^T \frac{\partial}{\partial \boldsymbol{u}} F(\boldsymbol{u}_h(t),0,0) = q_2(t)^T A(t) = -\dot{q}_2(t)^T.$$

To simplify (6.6) further, we note that (3.3) implies

$$\frac{\partial^2}{\partial \boldsymbol{u}^2} F^1(\boldsymbol{u}_h(t),0,0) \circ (\boldsymbol{z},\boldsymbol{z})$$

(6.7)

$$= \partial_x^2 f(x_h(t),0) \circ (z^1,z^1) + \partial_x^2 f(x_h(t),0) \circ (z^2,z^2)$$

and that (3.14) implies

$$\partial_\epsilon \overline{\boldsymbol{u}}(0,0) = \begin{pmatrix} 0 \\ \zeta_0 \end{pmatrix} \quad . \tag{6.8}$$

Applying (6.7) and (6.8) to (6.6) yields

$$\partial_\epsilon^2 H(0,0) \quad = \int_{-\infty}^{\infty} q_2(t)^T \{ \partial_x^2 f(x_h(t),0) \circ (\partial_\epsilon \hat{z}^1(0,0), \partial_\epsilon \hat{z}^1(0,0))$$

(6.9)

$$+ \partial_x^2 f(x_h(t),0) \circ (\zeta_0 + \partial_\epsilon \hat{z}^2(0,0), \zeta_0 + \partial_\epsilon \hat{z}^2(0,0)) \} dt \quad .$$

Next we compute $\partial_\epsilon \hat{z}^j(0,0)$ $(j=1,2)$. Recalling (6.2),

$$\begin{aligned} \mathcal{F}[\partial_\epsilon \boldsymbol{z}(0,0)] &= \mathcal{Q}(\boldsymbol{A}(\cdot) \partial_\epsilon \overline{\boldsymbol{u}}(0,0)) \\ &= \mathcal{Q} \begin{pmatrix} 0 \\ (A(t) - 2\nu_h D)\zeta_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ (A(t) - 2\nu_h D)\zeta_0 \end{pmatrix} \end{aligned}$$

that is,

$$\frac{d}{dt}\{\partial_\epsilon z^1(0,0)\} - A(t) \partial_\epsilon z^1(0,0) = 0$$

$$\frac{d}{dt}\{\partial_\epsilon z^2(0,0)\} - (A(t) - 2\nu_h D) \partial_\epsilon z^2(0,0) = (A(t) - 2\nu_h D)\zeta_0 \quad .$$

From this former equation we have

$$\partial_\epsilon z^1(0,0) = 0 \quad . \tag{6.10}$$

The latter equation is rewritten as

$$\frac{d}{dt}(\zeta_0 + \partial_\epsilon z^2(0,0)) - (A(t) - 2\nu_h D)(\zeta_0 + \partial_\epsilon z^2(0,0)) = 0 \tag{6.11}$$

But since $\partial_\epsilon \boldsymbol{z}(0,0) \in C_\gamma$, we see

$$\lim_{t\to-\infty} [\partial_\epsilon z^2(0,0)](t) = 0.$$

Hence we have a unique solution of (6.11)

$$\zeta_0 + [\partial_\epsilon z^2(0,0)] = p_0(t) \tag{6.12}$$

for which

$$\partial_\epsilon^2 H(0,0) = h_2 = \int_{-\infty}^{\infty} q_2(t)^T \partial_{xx} f(x_h(t),0) \circ (p_0(t), p_0(t)) dt \quad . \tag{6.13}$$

This is just the quantity (1.15) in Theorem 1.1.

From the above facts it follows that (for sufficiently small ϵ) on

$$\begin{aligned}(\mu,\nu) = {} & (\mu(\epsilon), \nu_0(\mu(\epsilon)) + \eta(\mu(\epsilon),\epsilon)) \\ = {} & (0,\nu_h) + \frac{\epsilon^2}{2}\left(\mu_2, \frac{\partial^2 \hat\eta}{\partial \epsilon^2}(0,0)\right) + o(\epsilon^2)\end{aligned}$$

there is a solution

$$\begin{aligned}\boldsymbol{u}_\epsilon(t) = {} & \boldsymbol{u}_h(t) + \overline{\boldsymbol{u}}(\mu(\epsilon),\epsilon) + [\hat{\boldsymbol{z}}(\mu(\epsilon),\epsilon)](t) \\ = {} & \boldsymbol{u}_h(t) + \epsilon\begin{pmatrix} 0 \\ \zeta_0 \end{pmatrix} + \epsilon[\partial_\epsilon \hat{\boldsymbol{z}}(0,0)](t) + O(\epsilon^2) \\ = {} & \boldsymbol{u}_h(t) + \epsilon\begin{pmatrix} 0 \\ \zeta_0 + [\partial_\epsilon \hat z^2(0,0)](t) \end{pmatrix} + O(\epsilon^2) \\ = {} & \begin{pmatrix} x_h(t) \\ \epsilon p_0(t) \end{pmatrix} + O(\epsilon^2)\end{aligned} \tag{6.14}$$

This expression is the desired one which appeared in (5.38). Thus we obtain the conclusion of the proof of Theorem 1.1.

7. An example. In this section we demonstrate that there are systems which exhibit the symmetry breaking homoclinic bifurcation of the main theorem. This is done by explicitly verifying all hypotheses in a model system. Moreover, for the model system, the twistedness of the bifurcating orbits can be determined for a range of coupling strengths.

First, we define

$$\dot x = f(x,\mu) := f_0(x) + \mu\begin{pmatrix} 0 \\ -x_2 \end{pmatrix} \quad , \qquad x = (x_1,x_2)^T \quad , \tag{7.1}$$

where

$$f_0(x) = \begin{pmatrix} x_2 \\ x_1 - \frac{3}{2}x_1^2 \end{pmatrix} \quad . \tag{7.2}$$

For $\mu = 0$, the system is logistic, hamiltonian and has a homoclinic solution (unique up to phase shift)

$$x_h(t) = \begin{pmatrix} x_{h1}(t) \\ x_{h2}(t) \end{pmatrix} = \begin{pmatrix} \mathrm{sech}^2(\frac{t}{2}) \\ -\sinh(\frac{t}{2})\mathrm{sech}^3(\frac{t}{2}) \end{pmatrix} \quad . \tag{7.3}$$

Note that $x_h(t)$ is homoclinic to $(0,0)^T$.

Having defined the (uncoupled) model equations, we begin by verifying (1.11). The linearization of (7.1) about $x_h(t)$ is

$$\dot{p} = A(t)p \quad , \quad A(t) = \begin{pmatrix} 0 & 1 \\ 1 - 3x_{h1}(t) & 0 \end{pmatrix} \quad . \tag{7.4}$$

Moreover, the bounded solution $p_1 = \dot{x}_h(t)$ has the asymptotics

$$p_1(t) = \dot{x}_h(t) \sim \begin{cases} \zeta_1 e^{-\alpha_1 t} = (-4,4)^T e^{-t} & , \quad t \to \infty \\ \zeta_2 e^{\alpha_2 t} = (4,4)^T e^{t} & , \quad t \to -\infty \end{cases} \tag{7.5}$$

The adjoint equation

$$\dot{q} = -A(t)^T q = \begin{pmatrix} 0 & -(1 - 3x_{h1}(t)) \\ -1 & 0 \end{pmatrix} \tag{7.6}$$

has a bounded solution

$$q_2(t) = \frac{1}{32} \begin{pmatrix} -\dot{x}_{h2}(t) \\ \dot{x}_{h1}(t) \end{pmatrix} \quad . \tag{7.7}$$

It is readily seen from (7.3),(7.5) and (7.7) that

$$\lim_{t\to\infty} q_2(-t)^T \dot{x}_h(t) e^{2t} = 1 \quad > 0 \quad ,$$

verifying (1.11).

Next, we compute ν_h and verify (1.7)-(1.8). For this we must first define the coupling matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d^2 \end{pmatrix} \quad . \tag{7.8}$$

For

$$A = \frac{\partial f}{\partial x}(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{7.9}$$

the matrix $A - 2\nu D$ has a zero eigenvalue when

$$\nu_h = \frac{1}{2d} \quad . \tag{7.10}$$

Indeed

$$A - 2\nu_h D = \begin{pmatrix} -1/d & 1 \\ 1 & -d \end{pmatrix}$$

has eigenvalues $\alpha_0 = 0$ and $-\alpha_3 = -(1/d + d)$ with corresponding eigenvectors

$$\zeta_0 = \begin{pmatrix} d \\ 1 \end{pmatrix}, \quad \zeta_3 = \begin{pmatrix} 1 \\ -d \end{pmatrix} \quad .$$

Since

$$(A - 2\nu_h D)^T \zeta_0^* = 0, \quad , \quad \zeta_0^* = (1+d^2)^{-1} \begin{pmatrix} d \\ 1 \end{pmatrix} \quad , \quad \zeta_0^{*T}\zeta_0 = 1 \quad ,$$

we can check the nondegeneracy condition (1.8)

$$\zeta_0^{*T} D\zeta_0 = (1+d^2)^{-1}(d \;\; 1)\begin{pmatrix} d \\ d^2 \end{pmatrix} = 2d^2/(1+d^2) > 0.$$

This condition assures the existence of bifurcating equilibria from $(x^1, x^2) = (0,0)$. For the coupled model presented in this section, all equilibria can be computed explicitly. Using the transformation (3.1), equilibria must satisfy

$$f(u+v,\mu) - 2\nu Dv = 0, \tag{7.11}$$

$$f(u-v,\mu) + 2\nu Dv = 0. \tag{7.12}$$

Noting asymmetric equilibria have $v \neq 0$ and taking advantage of the symmetry in these equations, the pitchfork eminating from $(x^1, x^2) = (0,0)$ is readily found in terms of (u, v):

$$(u_1, u_2, v_1, v_2) = \left(\frac{1}{3}(1-\Delta), 0, \pm\frac{1}{3}\sqrt{1-\Delta^2}, 2\nu v_1\right) \tag{7.13}$$

where

$$\Delta = 2\mu\nu + 4\nu^2 d^2 \quad . \tag{7.14}$$

For $\mu = 0$, it is easily seen that $v = 0$ at the previously computed bifurcation point $\nu = \nu_h = 1/(2d)$. That the bifurcating equilibria form a pitchfork (as depicted in Figure 3) is seen from the quadratic form defining v.

Next, we verify that h_1 is nonzero. Since

$$\partial_\mu f(x_h(t), 0) = \begin{pmatrix} 0 \\ -x_{h2}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -\dot{x}_{h1}(t) \end{pmatrix},$$

a direct calculation reveals

$$h_1 = \int_{-\infty}^{\infty} q_2(t)^T \partial_\mu f(x_h(t), 0)dt = -\frac{1}{32}\int_{-\infty}^{\infty} \dot{x}_{h1}^2(t)dt < 0 \quad .$$

The twistedness of the bifurcating homoclinic orbit now depends on the bounded solution of the linearized equation (1.9):

$$\dot{y} = (A(t) - 2\nu_h D)y = \begin{pmatrix} -1/d & 1 \\ 1 - 3x_{h1}(t) & -d \end{pmatrix} y \quad . \tag{7.15}$$

Letting this bounded solution be denoted by

$$p_0(t) = \begin{pmatrix} p_{01}(t) \\ p_{02}(t) \end{pmatrix} \quad ,$$

we prove at the end of this section the following lemma.

Lemma *For $d \in (0, d_-) \cup (d_+, \infty)$, the homoclinic solution is nontwisted and*

$$p_{01}(t) > 0, \quad \dot{p}_{01}(t) < 0 \quad (-\infty < t < \infty) \tag{7.16}$$

where $d_\pm = \frac{3\pm\sqrt{5}}{2}$. Furthermore, there exits a $\delta > 0$ so that for $d \in (d_-, d_- + \delta) \cup (d_+ - \delta, d_+)$ the homoclinic orbit is twisted.

Conclusion (7.16) is needed to compute the direction of the bifurcating homoclinic orbits in the (μ, ν)-plane. Since

$$\frac{1}{2}\partial_{xx} f(x_h(t), 0) \circ (p_0(t), p_0(t)) = -\begin{pmatrix} 0 \\ 3p_{01}^2(t) \end{pmatrix} \quad ,$$

we have

$$\begin{aligned} h_2 &= \int_{-\infty}^{\infty} q_2(t)^T \partial_{xx}^2 f(x_h(t),0) \circ (p_0(t),p_0(t))dt \\ &= -\frac{1}{6}\int_{-\infty}^{\infty} \dot{x}_{h1} p_{01}^2(t)dt \\ &= -\frac{1}{6}\left\{[x_{h1}(t)p_{01}^2(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x_{h1}(t)\cdot 2p_{01}(t)\dot{p}_{01}(t)dt\right\} \\ &= \frac{1}{3}\int_{-\infty}^{\infty} x_{h1}(t)p_{01}(t)\dot{p}_{01}(t)dt. \end{aligned}$$

Thus, for $d \in (0,d_-) \cup (d_+,\infty)$, (7.16) and $x_{h1}(t) > 0$ imply $h_2 < 0$. Moreover, $h_1h_2 > 0$ so that $\mu(\epsilon) < 0$. A similar proof for the twisted case remains elusive at present.

Proof of Lemma: To demonstrate twistedness (or nontwistedness) in the homoclinic solutions it suffices to examine the rotation in the bounded solution as $t \to \pm\infty$. Toward this end we set

$$p_{01} = r\cos\theta, \qquad p_{02} = r\sin\theta$$

in (7.15) yielding the system

$$\begin{aligned} &(7.17) \qquad \dot{r} = r\{-(1/d)\cos^2\theta - d\sin^2\theta + (2-3x_{h1}(t))\cos\theta\sin\theta\} \quad , \\ &(7.18) \qquad \dot{\theta} = (1/d-d)\cos\theta\sin\theta - \sin^2\theta + (1-3x_{h1}(t))\cos^2\theta \quad . \end{aligned}$$

Letting

$$w = \tan\theta - \frac{1}{2}(1/d-d) \quad ,$$

the equation (7.18) can be transformed into the Ricatti equation

$$(7.19) \qquad \dot{w} = \alpha^2 - 3x_{h1}(t) - w^2$$

where $\alpha = \frac{1}{2}(1/d+d)$.

Now consider the function $\bar{w}(t) = -3/2\tanh(t/2)$. One finds that

$$(7.20) \qquad \dot{\bar{w}} - \alpha^2 + 3x_{h1} + \bar{w}^2 = -\alpha^2 + \frac{9}{4}.$$

Thus if $\alpha > 3/2$, $\bar{w}$ is a lower solution for (7.19). Since $x_{h1}(t) > 0$ for all t, one finds that $w \equiv \alpha$ is an upper solution. It follows that the region in the t-w plane bounded below by the curve $w = -3/2\tanh(t/2)$ and above by $w = \alpha$ is a trapping region provided $\alpha > 3/2$. Therefore, the solution of (7.19) with $\lim_{t\to-\infty} w(t) = \alpha$ also must satisfy $\lim_{t\to\infty} w(t) = \alpha$. Since

$$\tan\theta(t) = w(t) + \frac{1}{2}(1/d-d)$$

we obtain

$$-d < \tan\theta(t) < 1/d, \qquad \lim_{t\to\pm\infty} \tan\theta(t) = 1/d$$

if $\alpha > 3/2$ which in terms of d corresponds to $d \in (0,d_-) \cup (d_+,\infty)$. Therefore, the homoclinic orbit is untwisted in this case.

Since, $-\pi/2 < \theta(t) < \pi/2$ it follows that

$$p_{01}(t) = r(t)\cos\theta(t) > 0 \quad .$$

Moreover, since $\tan(\theta(t)) < 1/d$,

$$\begin{aligned}\dot{p}_{01}(t) &= -(1/d)r(t)\cos\theta(t) + r(t)\sin\theta(t)\\ &= r(t)\cos\theta(t)(-1/d + \tan\theta(t)) < 0 \quad ,\end{aligned}$$

which concludes the proof of the first part of the lemma.

We see from (7.20) that when $\alpha = 3/2$, $\bar{w}$ is a solution of (7.19) satisfying

$$\lim_{t\to-\infty} w(t) = \alpha \quad ,$$

$$\lim_{t\to\infty} w(t) = -\alpha \quad .$$

Therefore if $d = \frac{3\pm\sqrt{5}}{2}$ the corresponding solution of (7.18) with $\lim_{t\to-\infty}\tan(\theta(t)) = 1/d$ satisfies $\lim_{t\to\infty}\tan(\theta(t)) = -d$, i.e., we have the degenerate case ($\hat{c}$ is in fact zero).

Next we consider the case of d slightly larger than d_-. The orbit for (7.18) corresponding to $\bar{w}$ acts as a separatrix for solutions that tend to $\arctan(1/d)$ and $\arctan(1/d) - \pi$ as $t \to \infty$ for $d = d_\pm$. Our goal is to show that as d increases past d_- the heteroclinic orbit $\bar{w}$ breaks so that the corresponding solution to (7.18) tends to $\arctan(1/d) - \pi$ resulting in a twisted orbit. To see this we first compactify the equation (7.18):

$$\dot{\theta} = (1/d - d)\cos\theta\sin\theta - \sin^2\theta + (1 - 3(1-\psi^2))\cos^2\theta \tag{7.21}$$

$$\dot{\psi} = \frac{1}{2}(1-\psi^2). \tag{7.22}$$

Since any solution to (7.22) with $\psi(0) = 0$ given by $\psi(t) = \tanh(t/2))$, it follows that the θ component of an solution to this system with $\psi(0) = 0$ is a solution of the non-autonomous equation (7.18). One checks that $(\arctan(1/d) + n\pi, 1)$ is a stable node for (7.21)- (7.22) while $(\arctan(-d) + n\pi, 1)$ and $(\arctan(1/d), -1)$ saddles for all $d > 0$ and each integer n.

We are interested in whether the unstable manifold of $(\arctan(1/d), -1)$ intersects the stable manifold of one of the saddles $(\arctan(-d) + n\pi, 1)$ or one of the attractors $(\arctan(1/d) + n\pi, 1)$ for a given d. If there is a saddle-node connection and n is even then the corresponding orbit is untwisted. If n is odd the orbit is twisted and if the connection is a saddle-saddle connection, we have the degenerate case.

We know that for $d \in (0, d_-)$ that there is a saddle-node connection with $n = 0$ and thus the orbit is untwisted. At $d = d_\pm$ we have the degenerate saddle-saddle connection, namely:

$$(\theta_0(t), \psi_0(t)) = (\arctan(-3/2\tanh(t/2) + \frac{1}{2}(1/d - d)), \tanh(t/2)).$$

If we can show that the center unstable manifold of the rest point $(\arctan(1/d), -1, d)$ for the system of (7.21)-(7.22) with the trivial flow $\dot{d} = 0$ appended intersects the center-stable manifold of $(\arctan(-d), 1, d)$ at $d = d_-$ it would follow that for d slightly greater than d_-, the orbit gives a connection between the saddle and the node $(\arctan(1/d) - \pi, 1)$. This is a standard Melnikov calculation. Following Jones [11], the intersection of the center-stable with the center unstable manifold is transverse provided the following integral is nonzero when $d = d_-$:

$$M(d) \equiv -(\frac{1}{d^2} + 1)\int_{-\infty}^{\infty} \dot{\psi}_0(t)\exp\left(-\int_0^t \text{trace[B(s)]ds}\right)\sin(\theta_0(t))\cos(\theta_0(t))dt \neq 0$$

where (θ_0, ψ_0) is the bounded solution of (7.21)- (7.22) and $B(t)$ is the Jacobian of the right-hand side of the system (7.21)-(7.22) evaluated at the heteroclinic orbit. After a long calculation we find that $M(d_-) = \frac{5}{384}\pi\left(31 + 21\sqrt{5}\right)$ and thus there is a $\delta > 0$ so that for $d \in (d_-, d_- + \delta)$ the orbit with $\theta(-\infty) = \arctan(1/d)$ tends to $\arctan(-1/d) - \pi$ and hence the orbit is twisted. A similar calculation shows that for $d \in (d_+, d_+ + \epsilon)$ the orbit is also twisted which completes the proof of the lemma.

8. Discussion. The results in this paper theoretically support a preliminary numerical study in a companion paper [13] where the bifurcating homoclinic orbits were computed using AUTO [2, 8]. There, a modified version of the muscle fiber model due to Murase [14] was investigated and the bifurcation curve in the (μ, ν)-plane for the asymmetric homoclinic orbits was found numerically. Though the theorem proven the present study guarantees such a bifurcation curve only local to $(\mu, \nu) = (0, \nu_h)$, the previous numerical study indicates these homoclinic solutions may persist well away from this parameter pair.

As mentioned in the introduction, the diffusively coupled systems examined here occur frequently as models of various biological phenonmenon. It is likely (though not proven) that periodic solutions bifurcate from the asymmetric homoclinic orbits in the main theorem. The stability of such solutions would be an important issue in the biological modelling community. In classical numerical studies of such systems, periodic solutions are found by continuation from Hopf points. In the companion paper, it was found in that model that such procedures could not be used to locate the homoclinic orbits bifurcating from $(\mu, \nu) = (0, \nu_h)$.

Finally we remark on a generalization of the main result. In this paper we used a 2-component system as the single model (1.1) since many important models addmitting a homoclinic bifurcation are described by 2-component systems. We, however, will be able to extend the study to a higher dimensional system of the single model provided it satisfies the conditions for exponential dichotomies and Melnikov function stated in section 2. We may also consider a diffusively coupled array model

$$\dot{x}^k = f(x^k, \mu) + \nu D(x^{k-1} - 2x^k + x^{k+1}), \quad k = 1, 2, \cdots, N, \quad x^0 = x^2, \quad x^{N+1} = x^{N-1},$$

and discuss the bifurcation similarly. This will be a future study.

Acknowledgements This study started when the first and the third authors were visiting School of Mathematics, Georgia Institute of Technology. They would like to take this opportunity to express their thanks to the school for the warm hospitality and special thanks to Professor Shui-Nee Chow for his very useful suggestions and information on the related studies. The first author also would like to express his acknowledgement to Professor Xiao-Bio Lin for sending him the paper prior to publication.

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