1. Let $P$ be a probability function and $A$, $B$ and $C$ be any 3 sets. Prove the following.

(a) If $A \subset B$ then $P(A) \leq P(B)$. (5 points)

(b) Suppose that events $A$, $B$, and $C$ are independent. Show that $A$ and $(B \cup C)$ are independent events. (5 points)

2. An urn contains two balls numbered 1 and 3. First a ball is drawn randomly from the urn, and then a fair coin is tossed the number of times as the number indicated on the drawn ball. Let $X$ equal the number of heads.

(a) Find the probability density function for $X$ (10pts).

(b) Find the mean and variance of $X$ (5 pts).

3. Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed $\text{Expon}(\beta)$ random variables.

(a) Show that $Z = X_1 + X_2 + \cdots + X_n$ has a $\text{Gam}(n, \beta)$ distribution. (5 points)

Now let $Z = X_1 + X_2 + \cdots + X_N$ with $N \sim \text{Geom}(p)$, i.e. $Z$ is a random sum of random variables. We have a hierarchical model with $Z|N \sim \text{Gam}(N, \beta)$ and $N \sim \text{Geom}(p)$.

(b) Find $E(Z)$. (5 points)

(c) Find $\text{Var}(Z)$. (5 points)

(d) Find the joint probability distribution of $(Z, N)$. (5 points)

(e) Find the marginal distribution of $Z$. (10 points)

4. Suppose a continuous random variable $X$ has a monotone increasing cumulative distribution function $F_X(x)$. Assume that the density function $f_X$ exists. Verify that the random variable $Y = F_X(X)$ has a uniform distribution over the interval $[0, 1]$. (5 points)

5. Let $X$ be a Poisson random variable with mean $\lambda$ and let $Y = e^X$. Find the probability distribution of $Y$. (10 points)

6. Let $X$ and $Y$ have joint density

$$f(x, y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the joint density of $U = X/Y$ and $V = X + Y$. (10 points)

(b) Are $U$ and $V$ independent? Justify your answer. (5 points)

(c) Find the marginal distributions of $U$ and $V$. (5 points)
7. The discrete random variable $X$ has 4 possible values: 0, 1, 2, and 3. Consider the 3 models given in the accompanying table of probabilities.

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.2</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>.5</td>
<td>.6</td>
</tr>
<tr>
<td>2</td>
<td>.1</td>
<td>.1</td>
</tr>
<tr>
<td>3</td>
<td>.2</td>
<td>.1</td>
</tr>
</tbody>
</table>

A Bayesian statistician places the following prior on $\Theta = (\theta_1, \theta_2, \theta_3)^T$: $\pi(\theta_1) = 0.3$, $\pi(\theta_2) = 0.4$, and $\pi(\theta_3) = 0.3$.

(a) Find the unconditional probability that $X = 1$. (5 pts)

(b) Find the posterior probability $\pi(\theta_3|X = 1)$. (5 pts)

8. Let $X_i$ be a random variable distributed as $N(i, i^2), i = 1, 2, 3$. Let $X_1, X_2, X_3$ be mutually independent.

(a) Find a random variable $U = U(X_1, X_2, X_3)$ such that $U$ has a chi-squared distribution with 3 degrees of freedom. (5 points)

(b) Find a random variable $V = V(X_1, X_2, X_3)$ such that $V$ has an F distribution with 1 numerator and 2 denominator degrees of freedom. (5 points)

9. Suppose $X_1, \ldots, X_n$ represent a random sample from a population with pdf

$$f_X(x|\theta) = \frac{2x}{\theta} \exp\left\{-\frac{x^2}{\theta}\right\} I_{(0,\infty)}(x), \quad \theta > 0.$$ 

Given: $E(X) = (1/2)\sqrt{\pi\theta}$ and $\text{Var}(X) = \theta \left(1 - \frac{\pi}{4}\right)$. The maximum likelihood estimator for $\theta$ is

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

(a) Find the maximum likelihood estimator for $E(X)$ and $\text{Var}(X)$. Justify your answer. (6 points)

(b) Is the MLE for $\theta$ a Uniform Minimum Variance Unbiased Estimator? Justify your answer. (8 pts)

(c) Find an exact $(1 - \alpha)100\%$ confidence interval for $\theta$. (Hint: $2X^2/\theta \sim \chi^2_2$. ) (10 pts)

10. State whether the statement is ALWAYS, SOMETIMES, or NEVER true. In each case justify your answer.

(a) An unbiased estimator is consistent. (5 points)

(b) A consistent estimator is unbiased. (5 points)

11. Let $X_1, X_2, \ldots, X_n$ be a random sample from $\text{N}(\mu, \theta)$.

(a) Find the Cramer-Rao Lower Bound (CRLB) for all unbiased estimators of $\theta$. (10 points)
(b) We know that the sample variance $S^2$ is unbiased for $\theta$.
   i. Find its variance. (Hint: Remember the distribution of $(n - 1)S^2/\theta$.) (5 points)
   ii. Compare the variance to the CRLB. (3 points)
   iii. Discuss, paying particular attention to what happens with large samples. (5 points)

12. Let $X_1, \cdots, X_n$ be iid Poisson($\lambda$).
   (a) Using the Central Limit Theorem find a large sample level $\alpha$ test of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. Be sure to identify the test statistic. (10 points)
   (b) Find a large sample level $\alpha$ LRT of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. Be sure to identify the test statistic. (10 points)

13. Let $X_1, \cdots, X_n$ be a random sample from a Uniform $(\theta, 2\theta)$ with $\theta > 0$.
   (a) Show that the smallest order statistic $X_{(1)}$ and the largest order statistic $X_{(n)}$ are a pair of joint minimal sufficient statistics for $\theta$. (10 points)
   (b) Find the MLE of $\theta$. Is it a function of a sufficient statistic? (10 points)