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An introduction to non-smooth analysis and geometry

Lecture 1: Motivation and Context

1. Introduction

The notion of a limit is the principle object in analysis. With it, we give precise meaning to real line, and hence to many other objects in mathematics (e.g., manifolds, vector spaces, etc.). One of the first things that one notices when trying to use calculus to study or solve problems is that *not all nice classes of objects are closed under taking the limit we wish to take!* For example, one can construct a sequence of smooth functions that converges uniformly to a function that is nowhere differentiable.

From a historical perspective, this first became a problem in the calculus of variations, i.e., in infimization problems. For example, consider the classical Plateau problem: given an unknotted Jordan curve in $\mathbb{R}^3$, find the surface filling the curve of minimal area. One might try to solve this problem by considering a sequence of nice smooth surfaces whose areas approach the minimal area, and then take some sort of limit. Simple examples show that finding the right sort of limit is very difficult, and that indeed the very notion of surface and area need to be extended beyond the classical smooth setting in order to find a "generalized minimizer". Once this is done, one studies *regularity theory* to determine how close this minimizer is to being "smooth".

Optimization problems for functions also generate similar issues. For example, consider the Dirichlet problem of classical complex analysis. Let Ω be a bounded open set in $\mathbb{C}$ with smooth boundary, and let $f\colon \partial\Omega \to \mathbb{R}$ be a smooth function. We wish to find a continuous function $u\colon \overline{\Omega} \to \mathbb{R}$ such that u is harmonic in Ω and $u|_{\partial\Omega} = f$. Dirichlet's original argument for the existence of such a function is the following: Stokes theorem can be used to show that a smooth function u with $u|_{\partial\Omega} = f$ is harmonic on Ω if and only if

$$\int_\Omega ||Du||^2 \, d\mathcal{L}^2 \leq \int_\Omega ||Dv||^2 \, d\mathcal{L}^2 \tag{1.1}$$

for all smooth function $v\colon \overline{\Omega} \to$ with the correct boundary values. Dirichlet considered the existence of such a minimizer to be obvious, but this is in fact far from the case since the space of functions we are minimizing over is *infinite dimensional*. Again, one must pass to a larger space of functions than "smooth functions with the correct boundary values" in order to gain the completeness properties needed to find a minimizer of a "generalized energy". Then, one can use a regularity theory to show that the minimizer is in fact smooth and harmonic.

Examples such as this are ubiquitous in analysis and geometry. Roughly speaking, the process is the following:

(1) one needs to understand how a certain class of functions or spaces behaves under a certain type of limit
(2) identify the appropriate closure of this class under the limit
(3) study which particular properties of the class persist under the limit
(4) see if these particular properties are enough to do what you wish to do.

When the objects in the class of interest are *nice metric spaces* (for example, Riemannian manifolds with some special properties), the natural type of limit arises from the *Gromov-Hausdorff distance*, which compares two metric spaces as abstract objects. Two (related) examples of the above process in this setting are:

- Gromov's compactness theorem: a set of compact Riemannian manifolds with diameter bounded above and Ricci curvature bounded below is pre-compact under Gromov-Hausdorff convergence.
- Perelman's work on the Poincaré conjecture: the homeomorphism type (same dimensional) Alexandrov spaces is preserved under Gromov-Hausdorff convergence

Other area of mathematics in which this process can be found includes:

- rigidity theory for symmetric spaces (e.g. Mostow rigidity)
- quasiconformal mapping theory
- geometric group theory
- diffusion geometry
- embedding theory for theoretical computer science

2. The focus of this course

The focus of this course will be the concept of differentiability in the non-smooth context.

The basic problem we would like to address is:

- The collection of differentiable functions on a smooth Riemannian manifold plays an important role in analysis and geometry.
- When the underlying smooth Riemannian manifold degenerates, the resulting limit space might not
  - be a manifold, or
  - possess a linear infinitesimal structure, i.e., tangent spaces,

  however, we would still like to have a class of "differentiable functions" on such limit spaces.

Our approach to this problem will be to identitfy a class of metric spaces, called *Poincaré inequality* spaces, that

- support a robust theory of differentiation for *Lipschitz* functions,
- include many smooth Riemannian spaces and important classical examples,
- is closed under Gromov-Hausdorff limits.

The basic outline of the course will be the following:

1. an introduction to the metric measure space setting,
2. classical differentiation results for Lipschitz functions on Euclidean spaces, i.e., Rademacher and Stepanov theorems,
3. an introduction to Poincaré inequality spaces and the associated Sobolev spaces of functions
4. Gromov Hausdorff convergence
5. Cheeger's theorem: Poincaré inequality spaces support measurable differentiable structures
6. applications.

The techniques that will be used are a mix of classical functional analysis, geometric measure, theory, and Riemannian geometry.

Some references that I have been using in preparing this course:

- *Lectures on Analysis on Metric Spaces*, Juha Heinonen
- *Sobolev Spaces on Metric Measure Spaces*, Heinonen, Koskela, Shanmugalingam, and Tyson
- *Sobolev met Poincaré*, Piotr Hajłasz
- *Differentiable Structures on Metric Measure Spaces: a Primer*, MacKay and Kleiner