Departments of Mathematics
Montana State University

Prof. Kevin Wildrick

An introduction to non-smooth analysis and geometry

Lecture 13: Gromov-Hausdorff Convergence

1. Hausdorff distance

Let (X, d) be a metric space, and consider the set

$$\mathcal{K}(X) := \{K \subseteq X : K \text{ is compact}\}.$$

Given $\delta > 0$ and a subset $S \subseteq X$, we denote the δ-neighborhood of S by

$$\mathcal{N}(S, \delta) := \bigcup_{x \in S} B_d(x, \delta).$$

Definition 1.1. The Hausdorff distance d_H on $\mathcal{K}(X)$ is defined by

$$d_H(K, L) = \inf\{\delta > 0 : K \subseteq \mathcal{N}(L, \delta) \text{ and } L \subseteq \mathcal{N}(K, \delta).$$

Note that the function d_H makes sense when evaluated on any pair of subsets of X; however, if K is bounded and L is unbounded, then $d_H(K, L) = \infty$. Moreover, $d_H(K, \overline{K}) = 0$ for any set K. For these reasons, we restrict our attention to $\mathcal{K}(X)$; it's an exercise to check that d_H is a metric on $\mathcal{K}(X)$, and with this metric $\mathcal{K}(X)$ is a compact metric space.

The Hausdorff metric is a standard tool in dynamics - if $f\colon X \to X$ is a contracting similarity, i.e.,

$$d_X(f(x), f(y)) = r d_X(x, y)$$

for some $0 < r < 1$, then f induces an L-Lipschitz map on $\mathcal{K}(X)$ for some $L < 1$. This implies that f has a unique fixed point in $\mathcal{K}(X)$.

2. Gromov-Hausdorff distance

Most metric spaces contain plenty of subsets that are distinct, but isometric to each other. For example, all the sets $[0, 1] \times \{y\}$ in $\mathbb{R}^2$ are isometric to each other, but have positive Hausdofff distance from each other.

Thus, if we'd like a way of measuring how far two metric spaces are from being isometric, we can't just use the Hausdorff distance (especially if the two metric spaces are not already sitting inside of some third space). The way around this is to consider all possible embeddings: For a pair of metric spaces (K, d_K) and (Z, d_Z), define.

$$\Phi(K; Z) = \{\phi\colon K \to Z : \phi \text{ is an isometry}\}.$$

Definition 2.1. Let (K, d_K) and (L, d_L) be compact metric spaces. Then the *Gromov-Hausdorff* distance between them is defined by

$$d_{GH}(K, L) := \inf\{d_H(\phi(K), \psi(L)) : \phi \in \Phi(K; Z), \psi \in \Phi(L; Z), Z \text{ is any metric space}\}.$$

Exericse 2.2.
- *Let K and L be compact metric spaces. Show that $d_{GH}(K, L) < \infty$. Hint: find a metric on the disjoint union of K and L that restricts to original metrics appropriately.*
- *Show that $d_{GH}(K, L)$ is zero if and only if K and L are isometric.*
- *Show that d_{GH} is a metric on the set of isometry classes of compact metric spaces.*

It is an important philosophical point that $d_{GH}(K, L)$ can be determined if one knows

$$d_{GH}(A, B)$$

for all finite subsets A of K and B of L. To see this, note that if $\{k_i\}_{i\in\mathbb{N}}$ is a countable dense subset of K, then

$$\lim_{n\to\infty} d_{GH}(\{k_i\}_{i=1}^n, A) = 0.$$

When the spaces K and L are not compact, we might run into problems - perhaps for every Z, and every $\phi \in \Phi(K; Z)$ and $\psi(L; Z)$, we have

$$d_H(\phi(K), \psi(L)) = \infty.$$

For example, suppose that K is a point and L is the real line. An even better example is a circle of radius i; letting i tend to infinity should result in something that looks more and more like a line, but the GH-distance will always stay infinite.

One way to avoid this is to give up on getting a metric on the space of all metric spaces, but instead settle for a notion of convergence.

Definition 2.3. A pointed metric space is a triple (X, d, x) where (X, d) is a metric space and x is a point of X.

We will only consider *proper* metric spaces here, i.e., spaces in which closed balls are compact. Our first attempt at defining Gromov-Hausdorff convergence of pointed metric spaces would certainly be the following. Let $\{(X_i, d_i, x_i)\}_{i\in\mathbb{N}}$ and $(X_\infty, d_\infty, x_\infty)$ be pointed metric spaces. Then $(X_i, d_i, x_i) \to (X_\infty, d_\infty, x_\infty)$ if for every $R > 0$,

$$\lim_{i\to\infty} d_{GH}(B_i(x_i, R), B_\infty(x_\infty, R)) = 0.$$

Unfortunately, this is a bit too restrictive: Let

$$(X_\infty, d_\infty, x_\infty) := (\{0, 1\}, d_{\mathbb{R}}, 0),$$

and

$$(X_i, d_i, x_i) = (\{-i^{-1}, 1 + i^{-1}\}, d_{\mathbb{R}}, i^{-1}).$$

Now, note that

$$\overline{B}_i(x_i, 1) = \{x_i\} \text{ and } \overline{B}_\infty(x_\infty, 1) = \{0, 1\}.$$

Thus, it is the case that

$$d_{GH}(\overline{B}_i(x_i, 1), \overline{B}_\infty(x_\infty, 1)) \not\to 0.$$

However, our intuition tell us we should have convergence of the pointed spaces. The "right" definition turns out to be the following:

Definition 2.4. Let $\{(X_i, d_i, x_i)\}_{i\in\mathbb{N}}$ and $(X_\infty, d_\infty, x_\infty)$ be pointed metric spaces. We say that $\{(X_i, d_i, x_i)\}_{i\in\mathbb{N}}$ *pointed Gromov-Hausdorff converges* to $(X_\infty, d_\infty, x_\infty)$ if there exists a sequence of mappings

$$\{\varphi_i \colon X_\infty \to X_i\}_{i\in\mathbb{N}}$$

such that

- $\varphi_i(x_\infty) = x_i$ for all $i \in \mathbb{N}$,
- for all $R > 0$,

$$\lim_{i\to\infty} \sup\{|d_i(\varphi_i(x), \varphi_i(y) - d_\infty(x, y)| : x, y \in B_\infty(x_\infty, R)\} = 0,$$

- for all $R > 0$ and all $\delta > 0$,

$$\lim_{i\to\infty} \sup\{d_i(y, \phi_i(B(x_\infty, R + \delta))) : y \in B_i(x_i, R)\} = 0.$$

Let us call the sequence of mappings $\varphi_i \colon X \to X_i$ a *Hausdorff approximation*.

Exericse 2.5. *Let $\{(X_i, d_i, x_i)\}_{i\in\mathbb{N}}$ and $(X_\infty, d_\infty, x_\infty)$ be pointed metric spaces, and assume that each space is compact.*

- *Suppose there is a constant $C \geq 1$ such that for all $i \in \mathbb{N}$,*
$$\operatorname{diam} X_i \leq C.$$
If $\{(X_i, d_i, x_i)\}_{i\in\mathbb{N}}$ pointed Gromov-Hausdorff converges to $(X_\infty, d_\infty, x_\infty)$, then
$$\lim_{i\to\infty} d_{GH}(X_i, X_\infty) = 0.$$
- *Suppose that*
$$\lim_{i\to\infty} d_{GH}(X_i, X_\infty) = 0.$$
Then $x_i \in X_i$ and x_∞ can be chosen so that $\{(X_i, d_i, x_i)\}_{i\in\mathbb{N}}$ pointed Gromov-Hausdorff converges to $(X_\infty, d_\infty, x_\infty)$.

Let's consider some examples:

- For an integer $i \geq 1$, let (X_i, d_i, x_i) be the pointed metric space $(\mathbb{S}^n, i \cdot d^3_{\mathbb{R}}, (1,0,0))$, and let $(X_\infty, d_\infty, x_\infty)$ be the pointed metric space $(\mathbb{R}^2, d_{\mathbb{R}^2}, (0,0))$. Then $\{(X_i, d_i, x_i)\}_{i\in\mathbb{N}}$ pointed Gromov-Hausdorff converges to $(X_\infty, d_\infty, x_\infty)$.
- Consider the symmetric, non-degenerate, bilinear form g_L on $T\mathbb{R}^3$ is given by
$$g_L(x,y,t) = \begin{pmatrix} 1+4y^2L & -4xyL & -2yL \\ -4xyL & 1+4x^2L & 2xL \\ -2yL & 2xL & L \end{pmatrix}.$$
From this form, we may define an inner-product
$$\langle v, w\rangle_L := v g_L w^\perp.$$
The matrix g_L is defined so that in the inner product $\langle\cdot,\cdot\rangle$, the vector fields
$$\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \frac{1}{\sqrt{L}}\frac{\partial}{\partial t}$$
form an orthonormal frame for $T\mathbb{R}^3$.

  In other words, we consider g_L as a Riemannian metric on $\mathbb{R}^3$. As we have discussed at the beginning of the course, this defines a metric space as follows: First, it defines a norm
$$||v||_L := \langle v, v\rangle^{1/2},$$
which in turn defines a notion of length: for a path $\gamma\colon [0,1] \to \mathbb{R}^3$,
$$\operatorname*{length}_L(\gamma) := \int_{[0,1]} ||\gamma'(s)||_L \, ds.$$
This gives rise to a path metric
$$d_L(p,q) := \inf\{\operatorname*{length}_L(\gamma) : \gamma\; p \rightsquigarrow q\}.$$
The pointed metric spaces $(\mathbb{R}^3, d_L, 0)$ converge (as $L \to \infty$) to $(\mathbb{H}, d_{cc}, 0)$. This is why the metric space $(\mathbb{H}, d_{cc})$ is called *sub-Riemannian*.
- For each $i \in \mathbb{N}$, let X_i be the standard $1/3$-Cantor set $\mathcal{C}$, let $x_i = 0$, and let $d_i = 3^i d_{\mathbb{R}}$. Then $\{(X_i, d_i, x_i)\}_{i\in\mathbb{N}}$ pointed Gromov-Hausdorff converges to $\bigcup_{k\in\mathbb{N}} \mathcal{C} + k$, with Euclidean metric and basepoint 0.

The notion of Gromov-Hausdorff convergence comes with a natural compactness result. Recall that a metric space (X,d) (not a metric measure space) is called D-doubling, $D \geq 0$, if each ball $B(x, 2r)$ can be covered by at most D-balls of radius r.

Exericse 2.6. *Show that if (X, d, μ) is a doubling metric measure space, then (X, d) is a doubling metric space.*

Theorem 2.7 (Gromov's Compactness Theorem). *Let $D \geq 0$, and suppose that $\{(X_i, d_i, x_i)\}$ is a sequence of D-doubling metric spaces, then there is a subsequence that pointed Gromov-Hausdorff converges to a complete D-doubling pointed metric space $(X_\infty, d_\infty, x_\infty)$.*

Proof. We give only a proof sketch. Fix $R > 0$ and $\epsilon > 0$. Conisder a set $N_i(R, \epsilon)$ inside $B(x_i, R)$ of maximal cardinality such that for any pair of distinct points x and y in $N_i(R, \epsilon)$,

$$d_i(x, y) \geq \epsilon.$$

Then the balls $\{B(x, \epsilon) : x \in N_i(R, \epsilon)\}$ cover $B(x_i, R)$. The uniform doubling condition implies that $\operatorname{card} N_i(R, \epsilon)$ is bounded above by a number $D(R, \epsilon)$ that is independent of i. By passing to subsequences and using a diagonalization argument, we can show that $\{(N_i(R, \epsilon), d_i)\}$ converges in the Gromov Hausdorff sense.

Now, take a sequence of ϵ tending to 0, and $R \to \infty$, and again passing to a subsequence, we get the result. □

3. Gromov-Hausdorff tangent spaces and tangent functions

Definition 3.1. A pointed metric space (X_0, d_0, p_0) is said to be a Gromov-Hausdorff tangent space of (X, d, p) if there is a sequence $\{r_i\}_{i \in \mathbb{N}}$ tending to 0 so that $(X, d/r_i, p)$ pointed Gromov-Hausdorff converges to (X_0, d_0, p_0).

Since rescaling preserves the doubling condition, Gromov's Compactness theorem implies that a doubling metric space has doubling Gromov-Hausdorff tangents at every point (subordinate to any sequence of scales). It is possible, but not extremely easy, to construct a doubling metric space where the collection of Gromov-Hausdorff tangents at a point contains more than one isometry class of metric spaces; in other words, the sequence of scales considered can make a difference!

Suppose that $X_i := \{(X, d/r_i, x)\}$ pointed Gromov-Hausdorff converges to (X_0, d_0, x_0), and let $f \colon X \to \mathbb{R}$ be an L-Lipschitz function. Let $\varphi_i \colon X_0 \to X_i$ be a Hausdorff approximation for the convergence of X_i to X_0. Define $f_i \colon X_0 \to \mathbb{R}$ by

$$f_i(x) = \frac{f(\varphi_i(x)) - f(\varphi_i(p))}{r_i}.$$

We wish to show that f_i converges uniformly on compact subsets of X_0.

To this end, let N_n be a $1/n$-net in $B_0(x_0, n)$. Since (X_0, d_0, x_0) is doubling, this is a finite set. If $x, y \in N_n$, then $d_0(x, y) \geq 1/n$, and so for $i \geq I(\epsilon, n)$,

$$|f_i(x) - f_i(y)| \leq L d_{X_i}(\varphi_i(x), \varphi_i(y)) \leq L(d_0(x, y) + n^{-2}) \leq L(1 + n^{-1}) d_0(x, y).$$

Thus, we may extend $(f_i)|_{N_n}$, $i \geq I(n)$, to an $(L + n^{-1})$-Lipschitz function F_i on X_0. Hence, by the Arzela-Ascoli theorem, the sequence of functions F_i has a subsequence that converges uniformly on compact subsets of X_0 to an L-Lipschitz function $f_0 \colon X_0 \to \mathbb{R}$. Let us pass to that subsequence. Then for any $n \in \mathbb{N}$, the restriction $f_0|_{N_n}$ is the limit of $\{f_i|_{N_n}\}$. Now, for any point $x \in X_0$, let $x_n \in N_n$ be such that $x_n \to x$. By continuity of f_0,

$$f_0(x) = \lim_{n \to \infty} f_0(x_n) = \lim_{n \to \infty} \lim_{i \to \infty} f_i(x_n).$$

Notice that by the above argument,

$$\lim_{n \to \infty} \lim_{i \to \infty} |f_i(x_n) - f_i(x)| \leq \lim_{n \to \infty} L(d_0(x_n, x) + n^{-2}) = 0,$$

and hence

$$\lim_{i \to \infty} f_i(x) = f_0(x).$$

Let us call f_0 a *tangent function* of f.

Thus, we can state the following consequence of Gromov's compactness theorem:

Corollary 3.2. *Let (X, d, p) be a pointed doubling metric space, and let $\{f_\alpha \colon X \to \mathbb{R}\}_{\alpha \in I}$ be a countable collection of Lipschitz functions on X. For any sequence of scales $r_i \searrow 0$, after passing to a subsequence, the pointed spaces $(X, d/r_i, p)$ point Gromov-Hausdorff converge to a pointed space (X_0, d_0, p_0), and for each $\alpha \in I$ there is a Lipschitz tangent function $f_{0,\alpha}$ of f_α so that*

$$f_{0,\alpha}(x) = \lim_{i\to\infty} \frac{f_\alpha(\varphi_i(x)) - f_\alpha(\varphi_i(p))}{r_i}.$$

Let us be more precise about the regularity of tangent functions of a Lipschitz function. For convenience, given a function $a \colon X \to Y$, a point $p \in X$, and a radius $r > 0$, denote

$$\operatorname{var} a(p, r) := \sup_{y \in B_X(x,r)} \frac{d_Y(a(x), a(y))}{r}.$$

With this notation,

$$\operatorname{Lip} a(p) = \limsup_{r\to 0} \operatorname{var} a(p, r), \text{ and } \operatorname{lip} a(p) = \liminf_{r\to 0} \operatorname{var} a(p, r).$$

Suppose that $f_0 \colon X_0 \to \mathbb{R}$ is a tangent function of a Lipschitz function $f \colon X \to \mathbb{R}$ at p. Then,

$$\begin{aligned}
\operatorname{var} f_0(p_0, r) &= \sup_{y \in B_{X_0}(p_0,r)} \frac{|f_0(p_0) - f_0(y)|}{r} \\
&= \lim_{i\to\infty} \sup_{y \in B_{X_0}(p_0,r)} \frac{|f(\varphi_i(p_0)) - f(\varphi_i(y))|}{r r_i} \\
&= \lim_{i\to\infty} \sup_{y \in B_X(p, r r_i)} \frac{|f(p) - f(y)|}{r r_i}.
\end{aligned}$$

Since this provides estimates only for one subsequence of scales, we acheive only the following estimates:

$$\operatorname{lip} f(p) \le \operatorname{var} f_0(p_0, r) \le \operatorname{Lip} f(p).$$

What's going on here is the following. We expect that the blow up of a Lipschitz function should be a something akin to a linear function. Linear functions on Euclidean space can in fact (and this is a not-so easy exercise) be detected by the condition that their variation (on any ball and any point) is constant. What we have acheived here is a weakening of that condition, but perhaps it is too much of a weakening.