Departments of Mathematics
Montana State University Fall 2015
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An introduction to non-smooth analysis and geometry

Lecture 14: The *Lip-lip* condition

1. The Lip-lip condition

Let (X, d, μ) be a doubling metric measure space, and suppose that $f_0\colon X_0 \to \mathbb{R}$ is a tangent function of a Lipschitz function $f\colon X \to \mathbb{R}$ at p. We have shown last time that

$$\operatorname{lip} f(p) \leq \operatorname{var} f_0(p_0, r) \leq \operatorname{Lip} f(p).$$

If we wish to have the limit function f_0 play the role of a "linear function", we will need to have more precise control.

Definition 1.1. A metric measure space (X, d, μ) has the Lip-*lip* condition if there is a constant $K \geq 1$ such that for every Lipschitz function $f\colon X \to \mathbb{R}$,

$$\operatorname{Lip} f(x) \leq K \operatorname{lip} f(x)$$

for μ-almost every $x \in X$.

If (X, d, μ) is a Lip-lip space, and f_0 is a tangent function of a Lipschitz function f on X at p, then for almost every choice of $p \in X$,

$$\frac{\operatorname{Lip} f(p)}{K} \leq \operatorname{var} f_0(p_0, r) \leq \operatorname{Lip} f(p).$$

This indicates that f_0 is similar to a linear function.

2. The Poincaré Inequality and the Lip-lip condition

Theorem 2.1. *Suppose that a complete doubling metric measure space (X, d, μ) admits a p-Poincaré inequality for some $p \geq 1$. Then there is a constant $K \geq 1$, depending only on the doubling and Poincaré inquality constants, so that (X, d, μ) is a K-Lip-lip space.*

We will need two lemmas:

Lemma 2.2. *A doubling metric measure space that supports a p-Poincaré inequalit for some $p \geq 1$ is* quasiconvex*, i.e., there is a constant $C_0 \geq 1$ such that for any pair of points $x, y \in X$, there is a path γ from x to y with*

$$\operatorname*{length}_d(\gamma) C d(x, y).$$

Proof. By iteration, it suffices to show the following: there is a constant $C_0 \geq 1$ such that for any pair of points $x, y \in X$, there is a path γ connecting a point of $A_x = B\left(x, \frac{d(x,y)}{4}\right)$ to a point of $A_y = B\left(y, \frac{d(x,y)}{4}\right)$.

A convenient notion in this proof is that of an ϵ-path: for $\epsilon > 0$, a sequence $x_0, \ldots x_k$ is called an ϵ-path if consecutive points have distance less than ϵ. The length of an ϵ-path is $\sum d(x_i, x_{i+1})$.

By the Arzela-Ascoli theorem (again!), it suffices to show that for any $\epsilon > 0$, there is an ϵ-path connecting a point of A_x to a point of A_y. To this end, for $K \geq 1$, define $f_K\colon X \to [0, K]$ by

$$f_K(z) = \min\{K, \text{the minimal length of an } \epsilon\text{-path connecting } z \text{ and } A_y\}.$$

Note that Then f_K is Iocally 1-Lipschitz and vanishes on A_y. Let

$$M = \inf_{z \in A_x} f_K(z), \text{ and } B = B\left(y, \frac{5d(x,y)}{4}\right).$$

Since μ is doubling,

$$\mu(A_x) \simeq \mu(A_y) \simeq \mu(B).$$

Either 0 or M is at least $M/2$ away from the average value $(f_K)_B$, so

$$⨍_B |f_K - (f_K)_B| \, d\mu \gtrsim M/2.$$

However, the Poincaré inequality implies

$$⨍_B |f_K - (f_K)_B| \, d\mu \lesssim Cd(x,y).$$

Thus $M \lesssim d(x,y)$. Choosing K large enough now yields the result. □

We note that the Lip-lip condition is invariant under bi-Lipschitz changes of metric: if $\Phi\colon (X,d_X) \to (Y,d_Y)$ is bi-Lipschitz, and (X,d_X) has the Lip-lip condition, then so does (Y,d_Y). The same is true for the completeness and the Poincaré inequality (with the pushed forward measure). Furthermore, a quasiconvex metric space (X,d) is bi-Lipschitz equivalent to its induced path-metric space (X,d_{length}). It is a nice exercise to show that a locally compact and complete path metric space is geodesic, i.e., any two points can be connected by a path whose length is the distance between the points.

The upshot of the previous paragraph is that we may assume, when proving Theorem 2.1, that (X,d,μ) is a geodesic metric measure space.

Recall that by Lusin's theorem, given a measurable function $g\colon (X,d,\mu) \to \mathbb{R}$ and a set $A \subseteq X$ of finite measure, for any $\epsilon > 0$ there is a subset $E \subseteq A$ with $\mu(A \backslash E) < \epsilon$ and a continuous function $\widetilde{g}\colon X \to \mathbb{R}$ of compact support in A with $\widetilde{g}|_E = g|_E$.

Let us say that a measurable function $g\colon X \to \mathbb{R}$ is approximately continuous at a point $x \in X$ if there is a set E for which $g|_E$ is continuous and x is a density point of E. By Lusin's theorem and the Lebesgue differentiation theorem, *every measureable function is approximately continuous almost everywhere!*. We will apply this to lip.

Proof of Theorem 2.1. As discussed above, we may assume without loss of generality that (X,d,μ) is a geodesic space.

Let $f\colon X \to \mathbb{R}$ be an L-Lipschitz function, and let $x \in X$, $r > 0$, and $y \in B(x,r)$. Let $\lambda \geq 1$. Then we may find a r/λ-chain of points $x = p_0, p_1, \ldots, p_n = y$ with $n \lesssim \lambda$. Let $B_i = B(p_i, r/\lambda)$. Then

$$|f(x) - f(y)| \leq \left| f(x) - ⨍_{B_0} f \, d\mu \right| + \left| \sum_{i=0}^{n-1} ⨍_{B_i} f \, d\mu - ⨍_{B_{i+1}} f \, d\mu \right| + \left| ⨍_{B_0} f \, d\mu - f(y) \right|.$$

First, note that

$$\left| f(x) - ⨍_{B_0} f \, d\mu \right| \leq ⨍_{B_0} L|x-z| \, d\mu(z) \leq \frac{Lr}{\lambda}.$$

A similar estimate holds for the correspodning expression involving y.

Now, let $B_i' = B(p_i, 3r/\lambda)$, so that $B_i \cup B_{i+1} \subseteq B_i'$. Then, by the doublign assumption,

$$|f_{B_i} - f_{B_i'}| \leq ⨍_{B_i} |f - f_{B_i'}| \, d\mu \lesssim ⨍_{B_i'} |f - f_{B_i'}| \, d\mu.$$

Thus, the Poincaré inequality (and the fact that $\operatorname{lip} f$ is an upper gradient) show that,

$$|f_{B_i} - f_{B'_i}| \lesssim \frac{r}{\lambda} \left(\fint_{CB'_i} (\operatorname{lip} f)^p \, d\mu \right)^{1/p}.$$

Now, suppose x is an approximate continuity point of $(\operatorname{lip} f)^p$, and let $\epsilon > 0$. We may find a set $E \subseteq B(x,r)$ and a continuous function $g \geq 0$ so that $(\operatorname{lip} f)^p = g$ on E, and $\mu(B(x,r) \backslash E) < \epsilon$. Thus, if $\lambda \geq 3C$,

$$\fint_{CB'_i} (\operatorname{lip} f)^p \, d\mu \leq \fint_{CB'_i} g \, d\mu + L^p \epsilon.$$

By the Lebesgue differentiation theorem, if $r < r(\epsilon, \lambda)$, then

$$\fint_{CB'_i} g \, d\mu \leq (\operatorname{lip} f(x))^p + \epsilon.$$

The upshot is that for $r < r(\epsilon, \lambda)$,

$$\left(\fint_{CB'_i} (\operatorname{lip} f)^p \, d\mu \right)^{1/p} \lesssim \operatorname{lip} f(x) + o(\epsilon).$$

Thus

$$|f_{B_i} - f_{B'_i}| \lesssim \frac{r}{\lambda} (\operatorname{lip} f(x) + o(\epsilon)).$$

Since

$$|f_{B_i} - f_{B_{i+1}}| \leq |f_{B_i} - f_{B'_i}| + |f_{B_{i+1}} - f_{B'_i}|,$$

we conclude that for $r < r(\epsilon, \lambda)$,

$$|f_{B_i} - f_{B_{i+1}}| \lesssim r \frac{\operatorname{lip} f(x) + o(\epsilon)}{\lambda}.$$

Hence, since $n \lesssim \lambda$, when $r < r(\epsilon, \lambda)$,

$$\frac{|f(x) - f(y)|}{r} \lesssim \frac{L}{\lambda} + \operatorname{lip} f(x) + o(\epsilon).$$

This impiles that

$$\operatorname{Lip} f(x) \lesssim \operatorname{lip} f(x).$$

□