Departments of Mathematics
Montana State University

Prof. Kevin Wildrick

An introduction to non-smooth analysis and geometry

Lecture 2: Metric measure spaces

In this lecture, we will discuss the basic structure of the spaces we are interested in studying. The key structures are drawn from the structure of Riemannian manifolds that have a chance of being preserved under limits: metric and measure.

1. The definitions and some examples

1.1. Measures.

In modern analysis, the role of measurability is much smaller than in a classical measure theory course. For this reason, we will adopt the following somewhat non-standard definition of the word "measure".

Definition 1.1. Let X be a topological space. A *measure* on X is a function $\mu\colon \mathcal{P}(X) \to [0,\infty]$ such that

- $\mu(\emptyset) = 0$,
- if $\{E_i\}_{i\in\mathbb{N}}$ is a countable collection of subset of X, then
$$\mu\left(\bigcup_i E_i\right) \leq \sum_{i\in\mathbb{N}} \mu(E_i),$$
- if $U \subseteq X$ is an open set and $E \subseteq X$ is any set, then
$$\mu(E) = \mu(E\cap U) + \mu(E\backslash U),$$
i.e., open sets are measurable,
- for any set $E \subseteq X$, there is a Borel set $U \supseteq E$ such that
$$\mu(E) = \mu(U),$$
i.e., μ is *Borel regular*.
- each point $x \in X$ has a neighborhood U with $\mu(X) < \infty$, i.e., μ is *locally finite*,
- there is a set $U \subseteq X$ such that $\mu(X) \neq 0$, i.e., μ is *non-trivial*.

In this course, we will not concern ourselves with measurability issues, as they are tedious and can in practice almost always be overcome with some standard techniques.

1.2. Metric spaces.

Definition 1.2. A metric space is a pair (X, d) where X is a set and $d\colon X\times X \to [0,\infty)$ is a function such that for all $x, y, z \in X$,

- $d(x, y) = 0$ if and only if $x = y$ (non-degeneracy),
- $d(x, y) = d(y, x)$ (symmetry),
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

For $x \in X$ and $r > 0$, we define the *open ball at x of radius r* by
$$B(x,r) := \{y \in X : d(x,y) < r\},$$
and for $r \geq 0$, the *closed ball at x of radius r* by
$$\overline{B}(x,r) := \{y \in X : d(x,y) \leq r\}.$$
The diameter of a set $E \subseteq X$ is defined
$$\operatorname{diam}(E) := \sup\{d(x,y) : x, y \in E\}.$$

We will use the convention that a metric space (X, d) is always equipped with the topology induced by the metric, namely, the smallest topology in which the open balls are open. Be careful, however, with the terminology: the topological closure of an open ball $B(x, r)$ might be different from $\overline{B}(x, r)$.

Metric spaces are *extremely* common in mathematics. We give here only a few examples - we'll see many more during the semester.

- Let V be a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Then
$$||v|| := (\langle v, v \rangle)^{1/2}$$
is a norm, and $d(v, w) = ||v - w||$ is a metric. The standard metric on $\mathbb{R}^n$ is defined in this way.
- Let V be a vector space equipped with a norm. Then d defined as before still defines a metric. For $1 \leq p \leq \infty$, the p-norm on $\mathbb{R}^n$ defines a metric in this way.
- Let M be a smooth manifold equipped with a smoothly varying inner product g on its tangent spaces. Let I be a compact interval in $\mathbb{R}$. For a continuous mapping $\gamma \colon I \to M$ (a *path*), define the *g-length* of γ by
$$\underset{g}{\operatorname{length}}(\gamma) = \int_I ||\gamma'(s)||_g^2 \, ds.$$
For points p and q of M, define
$$d_g(p, q) = \inf \underset{g}{\operatorname{length}}(\gamma),$$
where the infinimum is taken over all paths starting at p and ending at q. Then d_g is a metric on M. Unfortunately, both g and d_g are referred to as a Riemannian metric on M.
- In the previous example, we started with a notion of length and defined a metric. This can be reversed as well. Let (X, d) be any metric space, and let $\gamma \colon I \to X$ be a path in X. Then
$$\underset{d}{\operatorname{length}}(\gamma) := \sup \sum_{i=1}^{n} d(\gamma(s_{i-1}), \gamma(s_i))$$
where the infimum is taken over all partitions $s_0, \ldots, s_n$ of I. If (X, d) has the property that any two points can be connected by a path of finite length, then
$$\widetilde{d}(x, y) = \inf \underset{d}{\operatorname{length}}(\gamma)$$
is a metric on X, where the infimum is taken over all paths connecting x to y.
- If (X, d) is a metric space and $S \subseteq X$ is a subset, then $d|_{S \times S}$ defines a metric on S.
- If (X_1, d_1) and (X_2, d_2) are metric spaces, and $|| \cdot ||_p$ denote the p-norm on $\mathbb{R}^2$, then
$$d((x_1, x_2), (y_1, y_2) = ||(d_1(x_1, y_1), d_2(x_2, y_2))||_p$$
defines a metric on $X_1 \times X_2$.
- Let (X, d) be any metric space, and let $0 < \alpha < 1$. Then
$$d^\alpha(x, y) := (d(x, y))^\alpha$$
defines a metric on X called the α-*snowflaked* metric. If (X, d) Note that d^α is only sometimes a metric if $\alpha > 1$.
- Let (X, d) be a metric space, let Y be a topological space, and let $f \colon X \to Y$ be a homeomorphism. Then
$$d_Y(a, b) := d(f^{-1}(a), f^{-1}(b))$$

defines a metric on Y.

1.3. **Metric Measure Spaces.**

Definition 1.3. A *metric measure space* is a triple (X, d, μ) where (X, d) is a separable metric space, and μ is a measure on X.

1.4. **Examples of metric measure spaces.** Examples of metric measure spaces are found in nearly all parts of mathematics. Here we give just a few that indicate the historical motivations for this course come.

- Euclidean space $\mathbb{R}^n$ with its standard metric and Lebesgue measure,
- Any finite dimensional normed vector space with the associated metric and Lebesgue measure,
- Any orientend Riemannian manifold with its geodesic metric and the measure associated to its volume form,
- Sub-Riemannian manifolds and Carnot groups (definitions later)
- Finite dimensional simplicial complexes equipped with a path metric and the highest-dimensional Lebesgue measure
- plenty of "ad hoc" constructions

Perhaps the most general and important construction of a metric measure space comes from geometric measure theory.

Definition 1.4. Let (X, d) be a metric space, and let $0 \leq s < \infty$, and let $\delta > 0$. The *s-dimensional Hausdorff δ-content*, of a set $E \subseteq X$ is defined by

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{k \in \mathbb{N}} (\operatorname{diam} E_k)^s : E_k \subseteq X,\ \operatorname{diam} E_k < \delta,\ E \subseteq \bigcup_{k \in \mathbb{N}} E_k \right\}.$$

Then $\mathcal{H}^s_\delta(E)$ is a non-increasing function of δ (possibly a constant function with value ∞ or 0). Hence

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$$

is well defined; this is called the *s-dimensional Hausdorff measure of E.*

If the s-dimensional Hausdorff measure of (X, d) is finite and positive, then $(X, d, \mathcal{H}^s)$ is a metric measure space.

An important class of examples that arise in this way (but not in any of the other ways mentioned above) consists of *boundaries of hyperbolic groups.*