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# An introduction to non-smooth analysis and geometry

Lecture 5: Functions of Bounded Variation and Absolute Continuity

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## 1. Functions of Bounded Variation

The Lebesgue Differentiation Theorem for monotone functions also implies that if $f\colon \mathbb{R} \to \mathbb{R}$ is the *difference* of two monotone functions, then it is differentiable almost everywhere. The class of such functions turns out to be surprisingly large, and has a completely different description.

**Definition 1.1.** Let $[a, b]$ be a compact interval in $\mathbb{R}$, and let $f\colon [a, b] \to \mathbb{R}$ be a function. Given a partition

$$P = (t_0 = a < t_1 < \ldots < t_n = b)$$

of $[a, b]$, we define the *variation of $f$ on $P$* by

$$V(f, P) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|.$$

The *total variation of $f$ on $[a, b]$* is defined by

$$TV(f) = \sup_P V(f, P),$$

where the supremum is taken over all partitions of $[a, b]$. If the total variation of $f$ on $[a, b]$ is finite, we say that $f$ has *bounded variation* on $[a, b]$.

It's easy to check that all monotone and all Lipschitz functions are of bounded variation. It's also easy to see that the space of $BV$ functions on $[a, b]$ is a Banach space with the total variation as norm; there are many other ways to characterize BV functions.

The total variation of $f$ itself defines a non-decreasing function; we leave the following simple proposition to the reader.

**Proposition 1.2.** *Let $f$ be a function of bounded variation on $[a, b]$. Then for all $a \leq u \leq v \leq b$,*

$$TV(f|_{[a,v]}) = TV(f|_{[a,u]}) + TV(f|_{[u,v]}).$$

**Theorem 1.3.** *Let $f\colon [a, b] \to \mathbb{R}$ be a function. Then $f$ has bounded variation on $[a, b]$ if and only if it is the difference of two non-decreasing functions on $[a, b]$.*

*Proof.* First suppose that $f$ has bounded variation, and let $a \leq u \leq v \leq b$. Then, considering the partition $u < v$ of $[u, v]$, we see that

$$f(u) - f(v) \leq |f(u) - f(v)| \leq TV(f|_{[u,v]}) = TV(f|_{[a,v]}) - TV(f|_{[a,u]}).$$

This shows that

$$x \mapsto f(x) + TV(f|_{[a,x]})$$

is a non-decreasing function on $[a, b]$. Hence

$$f(x) = \big(f(x) + TV(f|_{[a,x]})\big) - TV(f|_{[a,x]})$$

is the difference of non-decreasing functions, as desired.

Now, suppose that $f$ is the difference of two non-decreasing functions $g, h$ on $[a, b]$, and let $a = t_0 < \ldots < t_n = b$ be a partition of $[a, b]$. Then

$$\begin{aligned}\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| &= \sum_{i=1}^{n} |(g(t_i) - g(t_{i-1})) - (h(t_i) - h(t_{i-1}))| \\ &\leq TV(g) + TV(h) \\ &= |g(b) - g(a)| + |h(b) - h(a)|.\end{aligned}$$

□

Hence, we have proved

**Corollary 1.4.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that is of bounded variation on each interval $[a, b] \subseteq \mathbb{R}$. Then $f$ is differentiable $m$-almost everywhere. In particular, if $f$ is Lipschitz, then $f$ is differentiable $m$-almost everywhere.*

## 2. Absolutely continuous functions and the fundamental theorem of calculus

The class of absolutely continuous functions plays an important role in analysis as precisely those functions for which the fundamental theorem of calculus holds.

**Definition 2.1.** Let $f\colon [a, b] \to \mathbb{R}$ be a function. We say that $f$ is *absolutely continuous* if for every $\epsilon > 0$, there is $\delta > 0$ such that if $\{(a_i, b_i)\}_{i=1}^{n}$ is a finite, disjointed collection of intervals in $[a, b]$ satisfying

$$\sum_{i=1}^{n} |b_i - a_i| < \delta,$$

then

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon.$$

Clearly absolutely continuous functions are continuous. Let us now establish that absolutely continuous functions are of bounded variation.

**Proposition 2.2.** *Let $f\colon [a, b] \to \mathbb{R}$ be an absolutely continuous function. Then it has bounded variation on $[a, b]$.*

*Proof.* By definition, we may find $\delta > 0$ such that if if $\{(a_i, b_i)\}_{i=1}^{n}$ is a finite, disjointed collection of intervals in $[a, b]$ satisfying

$$\sum_{i=1}^{n} |b_i - a_i| < \delta,$$

then

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < 1.$$

Clearly we may find a partition $t_0 = a < \ldots < t_n = b$ of $[a, b]$ such that $|t_i - t_{i-1}| < \delta$. Then by the definition of $\epsilon$,

$$TV(f) = \sum_{i=1}^{n} TV(f|_{[t_{i-1}, t_i]}) \leq n,$$

showing that $f$ has bounded variation. □

The previous proposition implies that absolutely continuous functions are differentiable almost everywhere. However, much more is true. The following theorem is the cornerstone of classical analysis.

**Theorem 2.3.** *Let $f\colon [a,b] \to \mathbb{R}$ be a function. Then the following are equivalent:*

(1) *$f$ is absolutely continuous,*
(2) *if $E \subseteq [a,b]$ is a set with $m(E) = 0$, then $m(f(E)) = 0$ as well,*
(3) *$f$ is differentiable $m$-almost everywhere, $f' \in L^1([a,b])$, and for each $x \in [a,b]$,*

$$f(x) - f(a) = \int_{[a,x]} f' \; dm.$$

The proof of this theorem is best given in the language of measure theory. We give a sketch and leave some detail to the reader.

*Proof.* The equiqualence of (1) and (2) is an elementary exercise in measure theory and we omit it. Similarly, that (3) implies (2) follows quickly from integration theory.

So, assume that (2) holds. We may (ignoring delicate measurability issues) define a measure $\mu$ on $[a,b]$ by

$$\mu(E) = m(f(E)).$$

By assumption, $\mu \ll m$, and so by the Radon-Nikodym theorem, there is a measurable function $\frac{d\mu}{dm}$ such that

$$\mu(E) = \int_E \frac{d\mu}{dm} \; dm.$$

For an interval $[x, x+h] \in (a,b)$, this impiles that

$$\frac{f(x+h) - f(x)}{h} = ⨍_{[x,x+h]} \frac{d\mu}{dm} \; dm.$$

A slight variant of the (general) Lebesgue differentiation theorem now shows that $f' = \frac{d\mu}{dm}$ at almost every point; the fundamental theorem of calculus formula now follows. ☐