STEINER'S FORMULA IN THE HEISENBERG GROUP

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ABSTRACT. Steiner's tube formula states that the volume of an ϵ -neighborhood of a smooth regular domain in $\mathbb{R}^n$ is a polynomial of degree n in the variable ϵ whose coefficients are curvature integrals (called also as quermassintegrals). We prove a similar result in the sub-Riemannian setting of the first Heisenberg group. In contrast to the Euclidean setting, we find that the volume of an ϵ-neighborhood with respect to the Heisenberg metric is an analytic function of ϵ that is generally not a polynomial. The coefficients of the series expansion can be explicitly written in terms of integrals of iteratively defined canonical polynomials of just five curvature terms.

CONTENTS

1. INTRODUCTION

Let us denote by $\Omega \subseteq \mathbb{R}^n$ a bounded regular domain in Euclidean space, and by Ω_ϵ its ϵ neighborhood with respect to the usual Euclidean metric. The celebrated Steiner's formula expresses the volume $\mathrm{vol}(\Omega_\epsilon)$ as a polynomial in ϵ

$$\mathrm{vol}(\Omega_\epsilon) = \sum_{k=0}^{n} a_k \epsilon^k, \tag{1.1}$$

where the coefficients a_k are the so called *quermassintegrals* of Ω.

This formula goes back to J. Steiner who proved it in two and three dimensional Euclidean spaces for convex polytopes. It has been generalized later by H. Weyl to the setting of arbitrary smooth submanifolds of $\mathbb{R}^n$. We refer the interested reader to the monograph of A. Gray [16] for an exhaustive overview of this subject, as well as [5]. A localized version of the above formula still holds even for non-smooth submanifolds as shown by H. Federer [8]. Recently these notions have

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been widely used to obtain new results concerning nonlinear PDEs and Sobolev inequalities, see e.g. [25], [20] and [13], and relative isoperimetric inequalities, see [12].

The purpose of this paper is to prove a similar result also in the sub-Riemannian setting of the first Heisenberg group $\mathbb{H}$. Indeed, it is well known that $\mathbb{H}$ can be endowed with its canonical left-invariant Carnot-Carathéodory metric, and therefore it is natural to search for a formula akin to (1.1), where the ϵ-neighborhood Ω_ϵ should be replaced by an ϵ-neighborhood with respect to the Carnot-Carathéodory metric (basic notation and results about the metric structure of the Heisenberg group can be found in Section 2). This interest is motivated by the recent progress in the geometric measure theory of Lie groups (e.g.[6, 18, 14, 1, 2, 17]). In the aforementioned papers, many tools of the Euclidean theory related to rectifiability and perimeter, such as co-area and divergence formulae, have been developed in the sub-Riemannian setting of non-commutative Lie groups. Nevertheless, the notions of higher order curvatures even in the simplest instance of $\mathbb{H}$ are still far from being fully understood. We hope this paper can provide some hints in this direction. For a general overview of these results we refer to the monograph [4].

Our approach is inspired by the work of R.C. Reilly [21], [22] which is based on expressing the coefficients a_k in Steiner's formula (1.1) in terms of integrals of iterated divergencies of the signed Carnot-Carathéodory distance function δ associated to Ω.

In our setting of the first Heisenberg group, instead of the full divergence, we consider the so called *horizontal divergence* of a *horizontal vector field* $X = u_1X_1 + u_2X_2$ where X_1 and X_2 are the canonical left-invariant horizontal vector fields in $\mathbb{H}$ and u_1 and u_2 are arbitrary smooth functions.

In this situation, the horizontal divergence of X is given by $\mathrm{div_H}\, X := X_1u_1 + X_2u_2$. If u is a smooth function in an open set of $\mathbb{H}$ we shall consider the *iterated horizontal divergencies* of u according to the relations:

$$\mathrm{div_H}^{(0)}\,\nabla_H u = 1,\ \mathrm{div_H}^{(i)}\,\nabla_{\mathrm{H}} u = \mathrm{div_H}\left((\mathrm{div_H}^{(i-1)}\,\nabla_H u)\cdot\nabla_H u\right),\quad i\geq 1,$$

where $\nabla_H u := (X_1u)X_1 + (X_2u)X_2$ is the *horizontal gradient of u*. With this notation our first statement reads as follows:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{H}$ be a bounded smooth domain with regular boundary and $Q \subseteq \mathbb{H}$ be a localizing set with the property that $\partial\Omega \cap Q$ is free from characteristic points. We denote by δ the signed Carnot-Carathédory distance function defined in a neighborhood of $\partial\Omega \cap Q$.*

For $\epsilon \geq 0$, let $\Omega_\epsilon \cap Q$ be a localized Heisenberg ϵ-neighborhood of Ω. Then the function $\epsilon \to vol(\Omega_\epsilon \cap Q)$ is real-analytic, and has a power series expansion given by

$$vol(\Omega_\epsilon\cap Q) = vol(\Omega\cap Q) + \sum_{i=1}^{\infty} a_i \frac{\epsilon^i}{i!},$$

where

$$a_i = \int_{\partial\Omega\cap Q} (\mathrm{div_H}^{(i-1)}\,\nabla_H\delta)\, d\mathcal{H}^2_{\mathbb{R}^3}.$$

The remarkable fact is, that although the $(i-1)^{\mathrm{st}}$ iterated divergence, $i \geq 1$, of a smooth function u contains, *a priori*, derivatives of order i, for the signed distance function δ this is not the case. It turns out that all coefficients a_i appearing in Theorem 1.1 are integrals of polynomials of certain second order derivatives of the function δ.

To simplify the notation for iterated applications of the vector fields $X_i, i = 1, 2$ we will use for $X_i(X_j)$ the notation X_{ij}.

The main result of our paper gives a precise recursive formula for the iterated divergences in terms of the following quantities:

$$A := \Delta_{\mathrm{H}}\delta := X_{11}\delta + X_{22}\delta \qquad B := -(4X_3\delta)^2 \qquad C := -4\left((X_1\delta)(X_{32}\delta) - (X_2\delta)(X_{31}\delta)\right)$$
$$D := 16X_{33}\delta \qquad E := 16\left((X_{31}\delta)^2 + (X_{32}\delta)^2\right)$$

Theorem 1.2. *Under the conditions of Theorem 1.1, the following relations hold:*

$$\mathrm{div_H}^{(1)}\,\nabla_H\delta = A, \qquad \mathrm{div_H}^{(2)}\,\nabla_H\delta = B + 2C$$
$$\mathrm{div_H}^{(3)}\,\nabla_H\delta = AB + 2D \qquad \mathrm{div_H}^{(4)}\,\nabla_H\delta = B^2 + 2BC + 2AD - 2E,$$

and for all $j \geq 2$*,*

$$\mathrm{div_H}^{(2j-1)}\,\nabla_H\delta = B^{j-2}\left(AB + 2(j-1)D\right), \tag{1.2}$$

$$\mathrm{div_H}^{(2j)}\,\nabla_H\delta = B^{j-2}\left(B^2 + 2BC + 2(j-1)(AD - E)\right). \tag{1.3}$$

Generally speaking, it is feasible to think that the integrals of iterated horizontal divergences appearing in the above expressions should carry important geometric information about the Heisenberg geometry of the domain Ω. In particular the expression

$$\mathrm{div_H}^{(1)}\,\nabla_H\delta = \Delta_{\mathrm{H}}\delta := X_{11}\delta + X_{22}\delta$$

is currently the accepted notion of the *horizontal mean curvature* of $\partial\Omega$, and indeed of the level sets $\{\delta = \epsilon\}$ for sufficiently small values of ϵ [1]. This notion of mean curvature plays a crucial role in the study of minimal surfaces in the Heisenberg group.

Analogously, the expression

$$\mathrm{div_H}^{(2)}\,\nabla_H\delta = -(4X_3\delta)^2 - 8\left((X_1\delta)(X_{32}\delta) - (X_2\delta)(X_{31}\delta)\right)$$

may provide a useful notion of the *horizontal Gauss curvature* of a surface in the Heisenberg group. While the above expression has not yet been investigated in depth, recent results [3], indicate however that this formula comes out as the limit of the sectional curvature of a surface in the Riemannian approximation of the Heisenberg group and gives an appropriate version of the Gauss-Bonnet theorem in the Heisenberg setting.

Some further information about the existing literature is now in order. In [9, Theorem 3.2], F. Ferrari provided a result analogous to Theorem 1.1, obtaining a recursive formula for the coefficients involved in the series based on the explicit computation of flow of the horizontal gradient of signed distance function; a track of this flow is called a *metric normal*, the theory of which has been developed in [1], [2], [26] . While this approach does not relate the iterated divergences to volume nor allow for localization, it is very effective in computing the volume function for explicit sets. For example, it was shown in [9] that Steiner's formula for a Carnot-Carathéodory ball in the Heisenberg group is indeed a polynomial of degree 4. We will use this approach in Section 5 to provide an example in which Steiner's formula is not a polynomial. More examples and details of this direct, geometric approach, were given in unpublished works [10] and [11]. Moreover, we point out that the relationship between Steiner's formula and an intrinsic, Heisenberg notion of Gauss curvature was also posited in [9], in the framework of that approach.

The paper is structured as follows: in Section 2 notations are fixed and background results of the Heisenberg calculus are recalled. Section 3 is devoted to a careful analysis of localizing sets and the link between the derivatives of the volume function and the integrals of the iterated divergences is established. In Section 4 we establish the recursive formulae stated in Theorem 1.2 and also give the proof of Theorem 1.1. In the final Section 5 we present examples of surfaces where the coefficients in Theorem 1.1 can be computed explicitly.

2. Notation and basic results

Given points $x = (x_1, x_2, x_3)$ and $x' = (x_1', x_2', x_3')$ in $\mathbb{R}^3$, the Heisenberg product is given by

$$x * x' = (x_1 + x_1', x_2 + x_2', x_3 + x_3' + 2(x_2 x_1' - x_1 x_2'),$$

defining the Heisenberg group $\mathbb{H}$. The corresponding Lie algebra is generated by the left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3},\ X_2 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3},\ X_3 = \frac{\partial}{\partial x_3}.$$

We employ this somewhat unusual notation for the readability of the computations to be made in Section 4.

The *horizontal distribution*

$$H\mathbb{H} := \operatorname{span}\{X_1, X_2\} \subseteq T\mathbb{R}^3$$

is equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathrm{H}}$ in which X_1 and X_2 form an orthonormal basis. This induces the *horizontal norm* $|| \cdot ||_{\mathrm{H}}$.

Since $[X_1, X_2] = -4X_3$, the horizontal distribution $H\mathbb{H}$ is non-integrable. It follows that any pair of points $x, x' \in \mathbb{H}$ can be connected by an absolutely continuous curve $\gamma\colon [0,1] \to \mathbb{R}^3$ with the property that $\gamma'(s) \in H_{\gamma(s)}\mathbb{H}$ for almost every $s \in [0,1]$; such a curve is called *horizontal*. Measuring the length of horizontal curves by using $|| \cdot ||_{\mathrm{H}}$ results in the Carnot-Carothéodory metric on $\mathbb{H}$, which is denoted

$$d_{CC}(x, x') := \inf \left\{ \int_\gamma ||\gamma'(s)||_{\mathrm{H}}\ ds : \gamma \text{ is a horizontal curve connecting } x \text{ to } x' \right\}.$$

The Haar measure on $\mathbb{H}$, the 3-dimensional Lebesgue measure $\mathcal{L}^3$, and the 4-dimensional Hausdorff measure $\mathcal{H}^4_{d_{CC}}$ of the metric d_{CC} all coincide up to a scaling. We will most often employ the Lebesgue measure.

Throughout this paper, we will work with the following **standing assumptions and notations**:

(1) We consider a fixed but arbitrary open set $\Omega \subseteq \mathbb{R}^3$ whose boundary $\partial\Omega$ is a $\mathcal{C}^4$-smooth surface.

(2) The signed distance of a point $g \in \mathbb{H}$ from $\partial\Omega$ is denoted by $\delta\colon \mathbb{H} \to [0, \infty)$, where

$$\delta(g) = \begin{cases} \operatorname{dist}_{CC}(g, \partial\Omega) & g \in \mathbb{H} \setminus \Omega, \\ -\operatorname{dist}_{CC}(g, \partial\Omega) & g \in \bar{\Omega}. \end{cases} \tag{2.1}$$

(3) The *characteristic set* of $\partial\Omega$ is defined by

$$\operatorname{char}(\partial\Omega) = \{ g \in \partial\Omega : T_g \partial\Omega = H_g \mathbb{H} \}.$$

This set is pathological from the perspective of the regularity of the distance function δ. Away from the characteristic set, the signed distance function δ has one degree of regularity less than $\partial\Omega$. Hence, we consider an arbitrary bounded, connected, and relatively open set $U_0 \subseteq \partial\Omega$ with the property that

$$\operatorname{dist}_{CC}(U_0, \operatorname{char}(\partial\Omega)) > 0.$$

A foundational result of [1] implies that there is a connected, bounded, and open set $U \subseteq \mathbb{R}^3$ that contains $\overline{U}_0$ and on which δ is $\mathcal{C}^3$-smooth.

(4) The Euclidean gradient field $\nabla\delta\colon U \to \mathbb{R}^3$ is normal (in the Euclidean sense) to the level set $\delta^{-1}(\epsilon)$ near any point of U. The projection of this vector field onto the horizontal distribution $H\mathbb{H}$ yields the *embedded horizontal normal* $N\colon U \to \mathbb{R}^3$ defined by

$$N = (X_1\delta)X_1 + (X_2\delta)X_2,$$

The basis of this work is the fact that the signed distance function satisfies the eikonal equation in the following sense:

$$\|N(g)\|_{\mathrm{H}} = 1, \qquad \text{for } \mathcal{L}^3\text{-almost every } g \in U. \tag{2.2}$$

In fact, the smoothness of δ implies that N is $\mathcal{C}^2$-smooth, and so (2.2) holds everywhere on U. It follows that the Euclidean gradient $\nabla\delta$ does not vanish at any point of U. Therefore, δ can be considered as a defining function for the level set $\delta^{-1}(\epsilon)$ near any point of U. This implies that these level sets are $\mathcal{C}^3$-smooth near any point of U.

(5) Given a differentiable function $\alpha\colon U \to \mathbb{R}$, we define the *horizontal gradient* of α to be the projection of the Euclidean gradient of α onto the horizontal distribution, i.e., $\nabla_{\mathrm{H}}\alpha\colon U \to \mathbb{R}^3$ is given by

$$\nabla_{\mathrm{H}}\alpha = (X_1\alpha)X_1 + (X_2\alpha)X_2.$$

Note that we have defined $\nabla_{\mathrm{H}}\alpha$ to be a vector field in $\mathbb{R}^3$, and not as the two-dimensional vector field $(X_1\alpha, X_2\alpha)\colon U \to \mathbb{R}^2$, as is often the case. In particular, $N = \nabla_{\mathrm{H}}\delta$. We will use both notations to denote this object: N will be employed when its role is geometric in nature, and $\nabla_{\mathrm{H}}\delta$ will be employed when its role is more analytic in nature.

(6) Let $V = aX_1 + bX_2\colon U \to \mathbb{R}^3$ be a differentiable vector field with values in the horizontal distribution. A key role in this paper is played by the *horizontal divergence* of V, which is defined by

$$\operatorname{div}_{\mathrm{H}} V = X_1 a + X_2 b.$$

3. The derivatives of the volume function

3.1. **The construction and properties of the localizing set Q.** If Ω is unbounded, the volume of its Heisenberg ϵ-neighborhood is infinite. To avoid this, we consider a localized version of the volume function. In the setting of $\mathbb{R}^3$, this can be done as follows. One assumes that $\overline{\Omega}$ has *positive reach*, meaning that there is a number $r > 0$ such that $\operatorname{dist}_{\mathbb{R}^3}(x, \overline{\Omega}) < r$, then there is a unique point $\pi_{\overline{\Omega}}(x) \in \overline{\Omega}$ of minimal distance to x. For each bounded Borel subset $Q \subseteq \mathbb{R}^3$ and $\epsilon \in [0, r)$, one considers the set

$$\mathcal{T}(Q, \overline{\Omega}, \epsilon) = \{x \in \mathbb{R}^3 : \operatorname{dist}_{\mathbb{R}^3}(x, \overline{\Omega}) \le \epsilon \text{ and } \pi_{\overline{\Omega}}(x) \in Q\}$$

and seeks a Taylor series expansion of the function

$$\epsilon \mapsto \mathcal{L}^3(\mathcal{T}(Q, \overline{\Omega}, \epsilon))$$

at $\epsilon = 0$.

The requirement that $\overline{\Omega}$ have positive reach is far weaker than our assumptions on Ω. Since we have assumed that $\partial\Omega$ is $\mathcal{C}^1$-smooth (even $\mathcal{C}^3$-smooth), we may view the set $\mathcal{T}(Q, \overline{\Omega}, \epsilon)$ above as the union of $\overline{\Omega} \cap Q$ and the tracks of $\partial\Omega \cap Q$ under the gradient flow of the Euclidean distance-to-$\partial\Omega$ function $\operatorname{dist}_{\mathbb{R}^3}(\cdot, \partial\Omega)$ for time ϵ. This gradient flow can also be viewed as the flow associated to the Euclidean outward-pointing normal to the level sets of $\operatorname{dist}_{\mathbb{R}^3}(\cdot, \partial\Omega)$. The volume of $\overline{\Omega} \cap Q$ is the constant term of the desired Taylor series.

It is (roughly) this later approach that we will adapt to the Heisenberg setting. Instead of considering an arbitrary Borel set $Q \subseteq \mathbb{R}^3$ for localization, we begin with any sufficiently regular set $B_0 \subseteq U_0 \subseteq (\partial\Omega \setminus \operatorname{char}(\partial\Omega))$ and define the localizing set Q to be the image of the flow associated with the embedded horizontal normal N. As mentioned in the introduction, this flow has been studied in depth as the *metric normal* in [1], [2], [26].

We now implement the above approach. For simplicity, we consider $B_0 \subseteq U_0 \subseteq \partial\Omega$ to be of the form

$$B_0 := \overline{B}_{\mathbb{R}^3}(p, r) \cap \partial\Omega,$$

where $p \in \mathbb{H}$ and $r > 0$ are chosen so that B_0 is homeomorphic to a closed disk; this situation generalizes easily to the situation that B_0 is the closure of any connected open subset of U_0 with Lipschitz boundary components.

Because of this simplification, we may parametrize the boundary ∂B_0 of B_0 with a single smooth function

$$\beta : [-\tau, \tau] \longrightarrow \partial B_0$$

for some $\tau > 0$.

The following proposition states that the flow of the embedded horizontal normal exists on any short time interval containing 0. This result can also be found in [1], we include its proof for completeness.

Proposition 3.1. *There exists $s_0 > 0$ such that for any $g_0 \in U_0 \subseteq \partial\Omega$, the Cauchy problem*

$$\begin{cases} \dot{\varphi}(s) = N(\varphi(s)) \\ \varphi(0) = g_0 \in U_0 \end{cases} \tag{3.1}$$

has a local solution $\varphi_{g_0} : [-s_0, s_0] \to U$ satisfying

$$d_{CC}(g_0, \varphi_{g_0}(\sigma)) \leq |\sigma|$$

for each $\sigma \in [-s_0, s_0]$. Moreover, if $g_0 \in \partial\Omega$, then

$$\delta(\varphi_{g_0}(\sigma)) = \sigma$$

for each $\sigma \in [-s_0, s_0]$.

Proof. Since N is $\mathcal{C}^2$-smooth on the open set U, the existence (and uniqueness) of a local solution to (3.1) follows from Peano-Picard theorem, even when g_0 is only assumed to be a point in U.

As $\bar{U}_0$ is a compact set contained in the open and bounded set $U \subseteq \mathbb{H}$, it holds that

$$\text{dist}_{CC}\,(U_0, \partial U) > 0.$$

We will prove that the maximal domain of the solution to (3.1) includes the interval $[-s_0, s_0]$ with $s_0 = \dfrac{\text{dist}_{CC}\,(U_0, \partial U)}{2}$. Towards a contradiction, let us suppose that there is a point $g_0 \in U_0$ such that the domain of the maximal solution $\varphi_{g_0}(\cdot)$ of (3.1) is an interval $(-T_1, T_2)$ that does not contain $[-s_0, s_0]$. We consider only the case that $0 < T_2 < s_0$; the other cases can be dealt with similarly. Since $||N||_{\mathrm{H}} = 1$, we see that for any point $\sigma \in (0, T_2)$,

$$\text{dist}_{CC}\,(\varphi_{g_0}(\sigma), U_0) \leq \int_0^\sigma ||N(\varphi_{g_0}(s))||_{\mathrm{H}}\, ds \leq \sigma < \frac{\text{dist}_{CC}\,(U_0, \partial U)}{2}.$$

Since $g_0 \in U_0 \subseteq U$ and N is bounded on every compact subset of U, it follows that

$$\varphi_{g_0}(T_2) := \lim_{\sigma \to T_2} \varphi_{g_0}(\sigma)$$

exists and it is contained in U. However, this implies that we can build a local solution to (3.1) with initial value $\varphi_{g_0}(T_2)$, contradicting the maximality of the interval of definition. This also shows that $\varphi_{g_0}([-s_0, s_0]) \subseteq U$. That

$$d_{CC}(g_0, \varphi_{g_0}(\sigma)) \leq |\sigma|$$

for all $\sigma \in [-s_0, s_0]$ follows from the fact that $||N||_{\mathrm{H}} = 1$.

To prove the final statement of the proposition, note that if $\sigma \in [-s_0, s_0]$, then the eikonal equation (2.2) implies that

$$(\delta \circ \varphi_{g_0})'(\sigma) = \langle \nabla\delta(\varphi_{g_0}(\sigma)), N(\varphi_{g_0}(\sigma)) \rangle = 1.$$

Integration now yields the desired equality. □

We define the *localizing set of depth* s_0 *generated by the set* $B_0 \subseteq U_0 \subseteq \partial\Omega$ by

$$Q := \{\varphi_g(s) : g \in B_0, |s| \le s_0\}. \tag{3.2}$$

For $0 \le \epsilon \le s_0$, we consider the *localized Heisenberg* ϵ*-tube*

$$\mathcal{T}_{\mathbb{H}}(B_0, \overline{\Omega}, \epsilon) := \{g \in Q : -s_0 \le \delta(g) \le \epsilon\} \tag{3.3}$$

and we seek to give a Taylor series expansion of the function

$$\epsilon \mapsto \mathcal{L}^3(\mathcal{T}_{\mathbb{H}}(B_0, \overline{\Omega}, \epsilon))$$

near $\epsilon = 0$. Note that we may write $\mathcal{T}_{\mathbb{H}}(B_0, \overline{\Omega}, \epsilon)$ as the disjoint union

$$\mathcal{T}_{\mathbb{H}}(B_0, \overline{\Omega}, \epsilon) = \big(\overline{\Omega} \cap Q\big) \cup Q_\varepsilon,$$

where

$$Q_\epsilon := \{g \in Q : 0 < \delta(g) < \varepsilon\}$$

is the track of B_0 under the flow of the embedded horizontal normal N for positive time ϵ. Hence, our localization procedure is in direct analogy with the Euclidean case. As in the Euclidean case, the volume of $\overline{\Omega} \cap Q$ will be the constant term of the desired Taylor series, and so we will be mostly concerned with estimating the volume of Q_ϵ.

The key tool in doing so is a version of the divergence theorem adapted to the structure of the Heisenberg group. For this we will need to identify the boundary of certain sets related to Q_ϵ. For $-s_0 < s < t < s_0$, denote

$$Q_{s,t} := \{g \in Q : s < \delta(g) < t\} = \delta^{-1}((s,t)) \cap Q,$$

so that $Q_\epsilon = Q_{0,\epsilon}$. We define the *initial boundary*, the *lateral boundary*, and the *final boundary* of $Q_{s,t}$ by

$$\begin{aligned}
\partial_i Q_{s,t} &:= \{\varphi_g(s), g \in B_0\} = \delta^{-1}(s) \cap Q, \\
\partial_l Q_{s,t} &:= \{\varphi_g(\epsilon) : g \in \partial B_0, s < \epsilon < t\}, \\
\partial_f Q_{s,t} &:= \{\varphi_g(t), g \in B_0\} = \delta^{-1}(t) \cap Q.
\end{aligned}$$

respectively. An elementary argument shows that

$$\partial(Q_{s,t}) = \partial_i Q_{s,t} \cup \partial_l Q_{s,t} \cup \partial_f Q_{s,t}. \tag{3.4}$$

In this setting, the Heisenberg divergence theorem takes a particularly simple form (c.f. [14, Corollary 7.7]), which we now describe. Define a vector field $\mu \colon \partial(Q_{s,t}) \to \mathbb{R}^3$ by

$$\mu(p) = \begin{cases} -\frac{\nabla\delta(p)}{||\nabla\delta(p)||_{\mathbb{R}^3}} & p \in \partial_i Q_{s,t}, \\ w(p) & p \in \partial_l Q_{s,t}, \\ \frac{\nabla\delta(p)}{||\nabla\delta(p)||_{\mathbb{R}^3}} & p \in \partial_f Q_{s,t}, \end{cases}$$

where $w \colon \partial(Q_{s,t}) \to \mathbb{R}^3$ is the Euclidean outward unit normal vector to $\partial(Q_{s,t})$. Then μ is the Euclidean unit outward-pointing normal vector field to $Q_{s,t}$. Denote its projection on to the horizontal distribution by μ_{H}, so that

$$\mu_{\mathrm{H}}(p) = \begin{cases} -\frac{N(p)}{||\nabla\delta(p)||_{\mathbb{R}^3}} & p \in \partial_i Q_{s,t}, \\ w_H(p) & p \in \partial_l Q_{s,t}, \\ \frac{N(p)}{||\nabla\delta(p)||_{\mathbb{R}^3}} & p \in \partial_f Q_{s,t}, \end{cases} \tag{3.5}$$

where w_{H} is the projection of w onto the horizontal distribution.

Now, let $V = aX_1 + bX_2 \colon U \to \mathbb{R}^3$ be any $\mathcal{C}^1$-smooth vector field with values in the horizontal distribution Applying the Euclidean divergence theorem and calculating, we see that

$$\begin{aligned}
\int_{Q_\epsilon} \operatorname{div}_{\mathrm{H}} V \, d\mathcal{L}^3 &= \int_{Q_{s,t}} \operatorname{div}\big((a, b, 2ax_2 - 2bx_1)\big) \, d\mathcal{L}^3 \\
&= \int_{\partial(Q_{s,t})} \langle (a, b, 2ax_2 - 2bx_1), \mu) \rangle_{\mathbb{R}^3} \, d\mathcal{H}^2_{\mathbb{R}^3} \qquad (3.6) \\
&= \int_{\partial(Q_{s,t})} \langle V, \mu_{\mathrm{H}} \rangle_{\mathrm{H}} \, d\mathcal{H}^2_{\mathbb{R}^3}.
\end{aligned}$$

The next result shows that on the lateral boundary, the vector w_{H} is perpendicular to the embedded horizontal normal N with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{H}}$. The analogous result in the Euclidean case is obvious.

Lemma 3.2. *Let $p \in \partial_l Q_{s,t}$. Then*

$$\langle N(p), w_{\mathrm{H}}(p) \rangle_{\mathrm{H}} = 0. \qquad (3.7)$$

Proof. Recall that we have already parametrized B_0 by the smooth function $\beta : [-\tau, \tau] \to \partial B_0$. Therefore, a parametrization of the lateral boundary $\partial_l Q_\epsilon$ is given by $\psi \colon [-\tau, \tau] \times (0, \epsilon)$, where

$$\psi(t, s) = \varphi_{\beta(t)}(s).$$

The Euclidean tangent space to $\partial_l Q_\epsilon$ at a point $p = (p_1, p_2, p_3) = \psi(t_0, s_0)$ is thus spanned by

$$v = (v_1, v_2, v_3) := \frac{\partial \psi}{\partial t}(t_0, s_0), \text{ and}$$

$$N(p) = \frac{\partial \psi}{\partial s}(t_0, s_0).$$

For convenience, denote $N(p) = n_1 X_1(p) + n_2 X_2(p)$. Taking the cross product of v and $N(p)$ now shows that $w(p)$ is a multiple of (w_1, w_2, w_3), where

$$\begin{aligned}
w_1 &= 2v_2(p_2 n_1 - p_1 n_2) - v_3 n_2 \\
w_2 &= -2v_1(p_2 n_1 - p_1 n_2) + v_3 n_1 \\
w_3 &= v_1 n_2 - v_2 n_1.
\end{aligned}$$

The projection $w_{\mathrm{H}}(p)$ of (w_1, w_2, w_3) onto the horizontal tangent space $H_p\mathbb{H}$ is given by

$$(w_1 + 2p_2 w_3) X_1(p) + (w_2 - 2p_1 w_3) X_2(p).$$

The result now follows from a simple calculation. ☐

Now, combining Lemma 3.2 with the expression in (3.5) and the divergence formula in (3.6) immediately proves the following version of the divergence theorem for a vector field that is a pointwise multiple of the embedded horizontal normal N.

Proposition 3.3. *Let $c \colon U \to \mathbb{R}$ be a $\mathcal{C}^1$-function. Then the vector field $cN : U \to \mathbb{R}^3$ satisfies*

$$\int_{Q_{s,t}} \operatorname{div}_{\mathrm{H}}(cN) \, d\mathcal{L}^3 = \int_{\partial_f Q_{s,t}} c \, ||\nabla \delta||^{-1}_{\mathbb{R}^3} \, d\mathcal{H}^2_{\mathbb{R}^3} - \int_{\partial_i Q_{s,t}} c \, ||\nabla \delta||^{-1}_{\mathbb{R}^3} \, d\mathcal{H}^2_{\mathbb{R}^3}$$

Although the above divergence formula involves the structure of the horizontal distribution, it is does not yet relate the Heisenberg surface measure $\mathcal{H}^3_{d_{CC}}$ on a level set of δ to the Lebesgue measure $\mathcal{L}^3$. However, away from the characteristic set, the Heisenberg surface measure is mutually absolutely continuous with Euclidean 2-dimensional surface measure, and the Radon-Nikodym derivative is precisely the Euclidean length of the gradient of δ. The relevant result is given in [6], which in our setting translates to the following: for any $s \in [-s_0, s_0]$ and any measurable function $F\colon Q \to \mathbb{R}$,

$$\int_{\delta^{-1}(s)\cap Q} F \, d\mathcal{H}^3_{d_{CC}} = \int_{\delta^{-1}(s)\cap Q} F \cdot \|\nabla\delta\|^{-1}_{\mathbb{R}^3} \, d\mathcal{H}^2_{\mathbb{R}^3} \tag{3.8}$$

Hence, we may record the following simple version of the Heisenberg divergence formula adapted to our setting:

Corollary 3.4. *Let c be as in Proposition 3.3, and let $-s_0 < s < t < s_0$. Then*

$$\int_{Q_{s,t}} \mathrm{div}_{\mathrm{H}}(cN) \, d\mathcal{L}^3 = \int_{\delta^{-1}(t)\cap Q} c \, d\mathcal{H}^3_{d_{CC}} - \int_{\delta^{-1}(s)\cap Q} c \, d\mathcal{H}^3_{d_{CC}}.$$

3.2. **Derivatives and iterated divergences.** In this section, we investigate the smoothness of the function

$$\epsilon \mapsto \mathcal{L}^3\left(\mathcal{T}_{\mathbb{H}}(B_0, \overline{\Omega}, \epsilon)\right)$$

on the interval $[0, s_0)$ by relating the function to the iterated divergences of δ.

For each integer $i \geq 0$, we define the function ${\mathrm{div}_{\mathrm{H}}}^{(i)} \delta\colon U \to \mathbb{R}$ by

$$\begin{aligned} {\mathrm{div}_{\mathrm{H}}}^{(0)} \nabla_H \delta &= 1, \\ {\mathrm{div}_{\mathrm{H}}}^{(i+1)} \nabla_H \delta &= \mathrm{div}_{\mathrm{H}}\left(\left({\mathrm{div}_{\mathrm{H}}}^{(i)} \nabla_H \delta\right) \cdot N\right). \end{aligned}$$

We will show that the sequence of derivatives $a^{(i)}\colon [0, s_0) \to \mathbb{R}$ inductively expressed by

$$\begin{aligned} a^{(0)}(\epsilon) &= \mathcal{L}^3\left(\mathcal{T}_{\mathbb{H}}(B_0, \overline{\Omega}, \epsilon)\right), \\ a^{(i+1)}(\epsilon) &= \begin{cases} \lim_{s \searrow 0} \frac{a^{(i)}(s) - a^{(i)}(0)}{s} & \epsilon = 0, \\ \lim_{s \to 0} \frac{a^{(i)}(\epsilon + s) - a^{(i)}(\epsilon)}{s} & \epsilon > 0, \end{cases} \end{aligned}$$

is indeed well-defined and can be expressed in terms of the above iterated divergences.

Theorem 3.5. *For each integer $i \geq 1$ and number $\epsilon \in [0, s_0)$, the limit $a^{(i)}(\epsilon)$ exists and is given by*

$$a^{(i)}(\epsilon) = \int_{\delta^{-1}(\epsilon)\cap Q} ({\mathrm{div}_{\mathrm{H}}}^{(i-1)} \nabla_H \delta) \, d\mathcal{H}^3_{d_{CC}} \tag{3.9}$$

We will need the following result regarding the continuity of integrals with respect to level sets of δ.

Lemma 3.6. *Let $F\colon Q \to \mathbb{R}$ be a continuous function. Then, for every $\epsilon \in (-s_0, s_0)$,*

$$\lim_{s \to \epsilon} \int_{\delta^{-1}(s)\cap Q} F \, d\mathcal{H}^2_{\mathbb{R}^3} = \int_{\delta^{-1}(\epsilon)\cap Q} F \, d\mathcal{H}^2_{\mathbb{R}^3} \tag{3.10}$$

Proof. Let $s \in (-s_0, s_0)$, and let $g \in \delta^{-1}(s) \cap Q$. Then $g = \varphi_{g_0}(s)$ for some $g_0 \in B_0$. Choose a local parametrization $\Phi : C_0 \longrightarrow B_0$ of B_0 near g_0. Then a local parametrization of $\delta^{-1}(s) \cap Q$ near g is given by $\omega_s(\cdot)\colon C_0 \longrightarrow \delta^{-1}(s) \cap Q$, where

$$\omega_s(\xi) = \varphi_{\Phi(\xi)}(s).$$

Therefore

$$\int_{\omega_s(C_0)} F(\lambda) d\mathcal{H}^2(\lambda) = \int_{C_0} F\left(\varphi_{\Phi(\xi)}(s)\right) |\mathcal{J}_{\omega_s}(\xi)| \, d\xi,$$

where $\mathcal{J}_{\omega_s}(\xi)$ is the Jacobian of ω_s at the point $\xi \in C_0$. Since Q is compact and ω_s is sufficiently smooth, the result follows by letting s tend to ϵ. ☐

Proof of Theorem 3.5. To begin, note that for $\epsilon > 0$,

$$a^{(0)}(\epsilon) = \mathcal{L}^3\left(\overline{\Omega} \cap Q\right) + \mathcal{L}^3(Q_\epsilon),$$

while $a^{(0)}(0) = \mathcal{L}^3\left(\overline{\Omega} \cap Q\right)$.

Using the Euclidean co-area formula, the continuity of the integral provided by Lemma 3.6, the mean value theorem, and the measure relationship given by (3.8), we see that for $\epsilon \geq 0$,

$$\begin{aligned}
\lim_{s\searrow 0} \frac{\mathcal{L}^3(Q_{\epsilon,\epsilon+s})}{s} &= \lim_{s\searrow 0} \frac{1}{s} \int_{Q_{\epsilon,\epsilon+s}} \frac{||\nabla\delta||_{\mathbb{R}^3}}{||\nabla\delta||_{\mathbb{R}^3}}\, d\mathcal{L}^3 \\
&= \lim_{s\searrow 0} \frac{1}{s} \int_\epsilon^{\epsilon+s} \left(\int_{\delta^{-1}(\sigma)\cap Q} ||\nabla\delta||_{\mathbb{R}^3}^{-1} d\mathcal{H}^2_{\mathbb{R}^3} \right) d\sigma \\
&= \int_{\delta^{-1}(\epsilon)\cap Q} ||\nabla\delta||_{\mathbb{R}^3}^{-1} d\mathcal{H}^2_{\mathbb{R}^3} \\
&= \int_{\delta^{-1}(\epsilon)\cap Q} d\mathcal{H}^3_{d_{CC}}. \qquad (3.11)
\end{aligned}$$

Setting $\epsilon = 0$ above now shows that

$$a^{(1)}(0) = \int_{\delta^{-1}(0)\cap Q} d\mathcal{H}^3_{d_{CC}} = \mathcal{H}^3_{d_{CC}}(\partial\Omega \cap B_0).$$

When $0 < \epsilon < s_0$, a similar argument for $s \nearrow 0$ now implies that

$$a^{(1)}(\epsilon) = \int_{\delta^{-1}(\epsilon)\cap Q} d\mathcal{H}^3_{d_{CC}}.$$

We now assume the inductive hypothesis that for an integer $i \geq 1$ and all $0 \leq \epsilon < s_0$,

$$a^{(i)}(\epsilon) = \int_{\delta^{-1}(\epsilon)\cap Q} {\mathrm{div}_{\mathrm{H}}}^{(i-1)}\, \nabla_{\mathrm{H}}\delta\, d\mathcal{H}^3_{d_{CC}}.$$

It follows that

$$\begin{aligned}
&\lim_{s\searrow 0} \frac{a^{(i)}(\epsilon+s) - a^{(i)}(\epsilon)}{s} \\
&\qquad = \int_{\delta^{-1}(\epsilon+s)\cap Q} ({\mathrm{div}_{\mathrm{H}}}^{(i-1)}\, \nabla_{\mathrm{H}}\delta)\, d\mathcal{H}^3_{d_{CC}} - \int_{\delta^{-1}(\epsilon)\cap Q} ({\mathrm{div}_{\mathrm{H}}}^{(i-1)}\, \nabla_{\mathrm{H}}\delta)\, d\mathcal{H}^3_{d_{CC}}.
\end{aligned}$$

Corollary 3.4 and the definition of the iterated divergences now yield

$$\lim_{s\searrow 0} \frac{a^{(i)}(\epsilon+s) - a^{(i)}(\epsilon)}{s} = \lim_{s\searrow 0} \int_{Q_{\epsilon,\epsilon+s}} {\mathrm{div}_{\mathrm{H}}}^{(i)}\, \nabla_{\mathrm{H}}\delta\ d\mathcal{L}^3.$$

Using the same argument that led to (3.11), we conclude that

$$\lim_{s\searrow 0} \frac{a^{(i)}(\epsilon+s) - a^{(i)}(\epsilon)}{s} = \int_{\delta^{-1}(\epsilon)\cap Q} {\mathrm{div}_{\mathrm{H}}}^{(i)}\, \nabla_{\mathrm{H}}\delta\ d\mathcal{H}^3_{d_{CC}}.$$

Using a similar line of reasoning in the case that $\epsilon > 0$ and $s \nearrow 0$, we now conclude that for all $\epsilon \in [0, s_0)$,

$$a^{(i+1)}(\epsilon) = \int_{\delta^{-1}(\epsilon)\cap Q} {\mathrm{div}_{\mathrm{H}}}^{(i)}\, \nabla_{\mathrm{H}}\delta\ d\mathcal{H}^3_{d_{CC}},$$

as desired. □

4. Calculating the iterated divergences

4.1. **A recursive formula.** Now that we have related the iterated divergences to the derivatives of the volume function, it behoves us to calculate the iterated divergences. The goal of this section is to show that all iterated divergences can be expressed using second-order derivatives of the signed distance function δ, although *a priori* $\mathrm{div_H}^{(i)} \nabla_H \delta$ involves i^{th}-order derivatives.

To simplify the notation in the coming computation, in this section we will denote the composition of vector fields $X_i(X_j)$ by X_{ij} and $X_i(X_j(X_k))$ by X_{ijk}, for $i,j,k \in \{1,2,3\}$. Products of vector fields will only be used once the vector fields have been applied to a function, namely δ. For example,

$$(X_i\delta)(X_{jk}\delta) = (X_i\delta)(X_j(X_k\delta)).$$

The following result gives a recursive formula for the iterated divergences in terms of the following quantities:

$$A := \Delta_{\mathrm{H}}\delta := X_{11}\delta + X_{22}\delta \qquad B := -(4X_3\delta)^2 \qquad C := -4\left((X_1\delta)(X_{32}\delta) - (X_2\delta)(X_{31}\delta)\right)$$

$$D := 16X_{33}\delta \qquad E := 16\left((X_{31}\delta)^2 + (X_{32}\delta)^2\right)$$

Theorem 4.1. *The following formulas hold:*

$$\mathrm{div_H}^{(1)} \nabla_H\delta = A, \qquad \mathrm{div_H}^{(2)} \nabla_H\delta = B + 2C$$

$$\mathrm{div_H}^{(3)} \nabla_H\delta = AB + 2D \qquad \mathrm{div_H}^{(4)} \nabla_H\delta = B^2 + 2BC + 2AD - 2E,$$

and for all $j \geq 2$,

$$\mathrm{div_H}^{(2j-1)} \nabla_H\delta = B^{j-2}\left(AB + 2(j-1)D\right), \tag{4.1}$$

$$\mathrm{div_H}^{(2j)} \nabla_H\delta = B^{j-2}\left(B^2 + 2BC + 2(j-1)(AD - E)\right) \tag{4.2}$$

The basic idea in the proof of Theorem 4.1 is to differentiate the eikonal equation to find relationships between various first, second, and third order derivatives of the signed distance function.

Lemma 4.2. *The following relations hold:*

$$\begin{cases} (X_1\delta)(X_{11}\delta) + (X_2\delta)(X_{12}\delta) = 0 \\ (X_1\delta)(X_{21}\delta) + (X_2\delta)(X_{22}\delta) = 0 \\ (X_1\delta)(X_{31}\delta) + (X_2\delta)(X_{32}\delta) = 0 \end{cases} \tag{4.3}$$

$$\begin{cases} (X_{11}\delta)(X_{22}\delta) = (X_{21}\delta)(X_{12}\delta) \\ (X_{11}\delta)(X_{32}\delta) = (X_{31}\delta)(X_{12}\delta) \\ (X_{21}\delta)(X_{32}\delta) = (X_{31}\delta)(X_{22}\delta) \end{cases} \tag{4.4}$$

$$\begin{cases} (X_{11}\delta)^2 + (X_1\delta)(X_{111}\delta) + (X_{12}\delta)^2 + (X_2\delta)(X_{112}\delta) = 0 \\ (X_{21}\delta)^2 + (X_1\delta)(X_{221}\delta) + (X_{22}\delta)^2 + (X_2\delta)(X_{222}\delta) = 0 \\ (X_{31}\delta)^2 + (X_1\delta)(X_{331}\delta) + (X_{32}\delta)^2 + (X_2\delta)(X_{332}\delta) = 0 \end{cases} \tag{4.5}$$

$$\begin{cases} (X_{11}\delta)(X_{31}\delta) + (X_1\delta)(X_{131}\delta) + (X_{12}\delta)(X_{32}\delta) + (X_2\delta)(X_{132}\delta) = 0 \\ (X_{21}\delta)(X_{31}\delta) + (X_1\delta)(X_{231}\delta) + (X_{22}\delta)(X_{32}\delta) + (X_2\delta)(X_{232}\delta) = 0 \end{cases} \tag{4.6}$$

$$X_{33}\delta = \frac{1}{4}(X_{321}\delta - X_{312}\delta) \tag{4.7}$$

$$\begin{cases} 8X_{32}\delta = -X_{122}\delta + X_{221}\delta \\ 8X_{31}\delta = -X_{112}\delta + X_{211}\delta \end{cases} \tag{4.8}$$

Proof. The eikonal equation (2.2) can be stated as

$$(X_1\delta)^2 + (X_2\delta)^2 = 1 \tag{4.9}$$

Therefore (4.3) follows from differentiating (4.9), by X_1, X_2, and X_3.

The equations in (4.3) show that the 3×2 matrix whose entries are X_{ij}, with $i = 1, 2, 3$ and $j = 1, 2$, has a non-trivial kernel. Therefore this matrix has rank at most 1, and the equations in (4.4) follow.

For (4.5), it is sufficient to differentiate the i^{th} equation of (4.3) by X_i, for $i = 1, 2, 3$. Similarly, the equations in (4.6) follow from differentiating the third equation of (4.3) by X_1 and X_2.

Finally, (4.7) and (4.8) hold because X_3 commutes with both X_1 and X_2, indeed:

$$X_{33}\delta = -\frac{1}{4}X_3(X_{12}\delta - X_{21}\delta) = \frac{1}{4}(-X_{312}\delta + X_{321}\delta)$$

and

$$\begin{aligned} 2X_{32}\delta = X_{32}\delta + X_{23}\delta &= -\frac{1}{4}(X_{12} - X_{21})X_2\delta + X_2\left(\frac{-X_{12}\delta + X_{21}\delta}{4}\right) \\ &= \frac{1}{4}(-X_{122}\delta + X_{212}\delta - X_{212}\delta + X_{221}\delta) \end{aligned}$$

The second equation of (4.8) is obtained in an analogous way. □

Recall that the embedded horizontal normal N is equal to the horizontal gradient of δ. Hence, by definition, for $i \geq 1$,

$$\mathrm{div_H}^{(i)}\,\nabla_H\delta = \mathrm{div_H}\left((\mathrm{div_H}^{(i-1)}\,\nabla_H\delta)\cdot\nabla_\mathrm{H}\delta\right) = (\mathrm{div_H}^{(i-1)}\,\nabla_H\delta)\Delta_\mathrm{H}\delta + \langle\nabla_\mathrm{H}(\mathrm{div}_{\mathbb{H}}^{(i-1)}\nabla_H\delta), \nabla_\mathrm{H}\delta\rangle,$$

where $\Delta_\mathrm{H}\delta$ is the horizontal Laplacian of δ, defined by

$$\Delta_\mathrm{H}\delta := \mathrm{div_H}(\nabla_\mathrm{H}\delta).$$

The second part of the above expression behaves very nicely. Indeed, the operator g defined by

$$g(\alpha) := \langle\nabla_\mathrm{H}\alpha, \nabla_\mathrm{H}\delta\rangle,$$

is linear and satisfies the Leibniz rule, i.e., given another differentiable function $\beta\colon U \to \mathbb{R}$,

$$\begin{aligned} g(\alpha + \beta) &= g(\alpha) + g(\beta), \\ g(\alpha\,\beta) &= g(\alpha)\beta + \alpha g(\beta). \end{aligned}$$

With this notation in hand, we may write

$$\mathrm{div_H}^{(i)}\,\nabla_H\delta = A\left(\mathrm{div_H}^{(i-1)}\,\nabla_H\delta\right) + g\left(\mathrm{div_H}^{(i-1)}\,\nabla_H\delta\right). \tag{4.10}$$

Lemma 4.3. *The following relations hold*

$$g(1) = 0 \tag{4.11}$$

$$g(A) = B + 2C - A^2 \tag{4.12}$$

$$g(B) = 0 \tag{4.13}$$

$$g(C) = D - AC \tag{4.14}$$

$$g(D) = -E \tag{4.15}$$

$$g(E) = -2AE + 2CD \tag{4.16}$$

Proof. The equality (4.11) follows from the Leibniz rule. For (4.12), we calculate

$$\begin{aligned}
g(A) &= \langle \nabla_{\mathrm{H}}(X_{11}\delta + X_{22}\delta), \nabla_{\mathrm{H}}\delta\rangle = (X_1\delta)\,(X_{111}\delta + X_{122}\delta) + (X_2\delta)\,(X_{211}\delta + X_{222}\delta) \\
&\overset{(4.5)}{=} -(X_{11}\delta)^2 - (X_{12}\delta)^2 - (X_2\delta)(X_{112}\delta) \\
&\quad - (X_{21}\delta)^2 - (X_1\delta)(X_{221}\delta) - (X_{22}\delta)^2 + (X_1\delta)(X_{122}\delta) + (X_2\delta)(X_{211}\delta) \\
&= -\left(X_{11}^2\delta + X_{22}^2\delta + 2(X_{12}\delta)(X_{21}\delta)\right) - \left(X_{21}^2\delta + X_{12}^2\delta - 2(X_{12}\delta)(X_{21}\delta)\right) \\
&\quad - (X_1\delta)(-X_{122}\delta + X_{221}\delta) + (X_2\delta)(-X_{112}\delta + X_{211}\delta) \\
&\overset{(4.4),(4.8)}{=} -(X_{11}\delta + X_{22}\delta)^2 - (-4X_3\delta)^2 - 8(X_1\delta)(X_{32}\delta) + 8(X_2\delta)(X_{31}\delta) \\
&= -A^2 + B + 2C.
\end{aligned}$$

For (4.13),

$$g(B) = -\langle \nabla_{\mathrm{H}}(X_3\delta)^2, \nabla_{\mathrm{H}}\delta\rangle = -2(X_1\delta)(X_3\delta)(X_{13}\delta) - 2(X_2\delta)(X_3\delta)(X_{23}\delta) \overset{(4.3)}{=} 0.$$

For (4.14),

$$\begin{aligned}
g(C) &= -4\left(\langle \nabla_{\mathrm{H}}((X_1\delta)(X_{32}\delta)), \nabla_{\mathrm{H}}\delta\rangle - \langle \nabla_{\mathrm{H}}((X_2\delta)(X_{31}\delta)), \nabla_{\mathrm{H}}\delta\rangle\right) \\
&= -4(X_{32}\delta)\left((X_1\delta)(X_{11}\delta) + (X_2\delta)(X_{21}\delta)\right) + 4(X_{31}\delta)\left((X_1\delta)(X_{12}\delta) + (X_2\delta)(X_{22}\delta)\right) \\
&\quad - 4(X_1\delta)\left((X_1\delta)(X_{132}\delta) + (X_2\delta)(X_{232}\delta)\right) + 4(X_2\delta)\left((X_1\delta)(X_{131}\delta) + (X_2\delta)(X_{231}\delta)\right) \\
&\overset{(4.3),(4.6)}{=} -16(X_3\delta)\left((X_1\delta)(X_{31}\delta) + (X_2\delta)(X_{32}\delta)\right) \\
&\qquad - 4(X_1\delta)\left((X_1\delta)(X_{123}\delta - X_{213}\delta) - (X_{21}\delta)(X_{31}\delta) - (X_{22}\delta)(X_{32}\delta)\right) \\
&\qquad + 4(X_2\delta)\left((X_2\delta)(X_{213}\delta - X_{123}\delta) - (X_{11}\delta)(X_{31}\delta) - (X_{12}\delta)(X_{32}\delta)\right) \\
&\overset{(4.7)}{=} 16\left((X_1\delta)^2 + (X_2\delta)^2\right)(X_{33}\delta) \\
&\quad + 4(X_1\delta)\left((X_{21}\delta)(X_{31}\delta) + (X_{22}\delta)(X_{32}\delta)\right) - 4(X_2\delta)\left((X_{11}\delta)(X_{31}\delta) + (X_{12}\delta)(X_{32}\delta)\right) \\
&\quad + 4(X_1\delta)(X_{11}\delta)(X_{32}\delta) - 4(X_1\delta)(X_{11}\delta)(X_{32}\delta) + 4(X_2\delta)(X_{22}\delta)(X_{31}\delta) - 4(X_2\delta)(X_{22}\delta)(X_{31}\delta) \\
&= D + 4\left[(X_1\delta)(X_{32}\delta) - (X_2\delta)(X_{31}\delta)\right](X_{11}\delta + X_{22}\delta) \\
&\qquad - 4(X_{32}\delta)\left[(X_1\delta)(X_{11}\delta) + (X_2\delta)(X_{12}\delta)\right] + 4(X_{31}\delta)\left[(X_1\delta)(X_{21}\delta) + (X_2\delta)(X_{22}\delta)\right] \\
&\overset{(4.3)}{=} D - AC.
\end{aligned}$$

For (4.15),

$$g(D) = 16\langle \nabla_{\mathrm{H}}(X_{33}\delta), \nabla_{\mathrm{H}}\delta\rangle = 16\left[(X_1\delta)(X_{133}\delta) + (X_2\delta)(X_{233}\delta)\right] \overset{(4.5)}{=} -E.$$

For (4.16),

$$
\begin{aligned}
g(E) &= 16\langle \nabla_{\mathrm{H}} \left((X_{31}\delta)^2 + (X_{32}\delta)^2\right), \nabla_{\mathrm{H}}\delta\rangle \\
&= 32(X_1\delta)\left[(X_{31}\delta)(X_{131}\delta) + (X_{32}\delta)(X_{132}\delta)\right] + 32(X_2\delta)\left[(X_{31}\delta)(X_{231}\delta) + (X_{32}\delta)(X_{232}\delta)\right] \\
&\overset{(4.6)}{=} 32(X_{31}\delta)\left[-(X_{11}\delta)(X_{31}\delta) - (X_{12}\delta)(X_{32}\delta) - (X_2\delta)(X_{123}\delta - X_{213}\delta)\right] \\
&\qquad + 32(X_{32}\delta)\left[-(X_{21}\delta)(X_{31}\delta) - (X_{22}\delta)(X_{32}\delta) + (X_1\delta)(X_{123}\delta - X_{213}\delta)\right] \\
&\overset{(4.7)}{=} -128(X_{33}\delta)\left[(X_1\delta)(X_{32}\delta) - (X_2\delta)(X_{31}\delta)\right] - 32(X_{11}\delta)(X_{31}\delta)^2 - 32(X_{22}\delta)(X_{32}\delta)^2 \\
&\quad - 32(X_{31}\delta)(X_{32}\delta)\left[X_{12}\delta + X_{21}\delta\right] + 32(X_{11}\delta)(X_{32}\delta)^2 - 32(X_{11}\delta)(X_{32}\delta)^2 \\
&\quad + 32(X_{22}\delta)(X_{31}\delta)^2 - 32(X_{22}\delta)(X_{31}\delta)^2 \\
&= 2CD - 32\left(X_{11}\delta + X_{22}\delta\right)\left[(X_{31}\delta)^2 + (X_{32}\delta)^2\right] + 32(X_{31}\delta)\left[(X_{22}\delta)(X_{31}\delta) - (X_{21}\delta)(X_{32}\delta)\right] \\
&\qquad + 32(X_{32}\delta)\left[(X_{11}\delta)(X_{32}\delta) - (X_{12}\delta)(X_{31}\delta)\right] \\
&\overset{(4.4)}{=} 2CD - 2AE.
\end{aligned}
$$

□

Proof of Theorem 4.1. The first four iterated divergences are easy to calculate using (4.10).

We proceed by induction on j, having already proven the desired result when $j = 2$. Assume that (4.1) and(4.2) hold for some $j \geq 2$. Now,

$$
\begin{aligned}
{\mathrm{div}_{\mathrm{H}}}^{(2(j+1)-1)} \nabla_H\delta &= {\mathrm{div}_{\mathrm{H}}}^{(2j+1)} \nabla_H\delta = A\, {\mathrm{div}_{\mathrm{H}}}^{(2j)} \nabla_H\delta + g({\mathrm{div}_{\mathrm{H}}}^{(2j)} \nabla_H\delta) \\
&\overset{(4.2)}{=} AB^{j-2}\left({\mathrm{div}_{\mathrm{H}}}^{(4)} \nabla_H\delta + 2(j-2)(AD - E)\right) \\
&\quad + B^{j-2} g\left({\mathrm{div}_{\mathrm{H}}}^{(4)} \nabla_H\delta + 2(j-2)(AD - E)\right) \\
&= AB^{j-2}\left(B^2 + 2BC + 2AD - 2E + 2(j-2)(AD - E)\right) \\
&\quad + B^{j-2} g\left(B^2 + 2BC + 2AD - 2E + 2(j-2)(AD - E)\right) \\
&= AB^j + 2AB^{j-1}C + 2(j-1)A^2B^{j-2}D - 2(j-1)AB^{j-2}E \\
&\quad + B^{j-2} g\left(B^2 + 2BC + 2(j-1)(AD - E)\right) \\
&= AB^j + 2AB^{j-1}C + 2(j-1)A^2B^{j-2}D - 2(j-1)AB^{j-2}E \\
&\quad + B^{j-2}\left(2BD - 2ABC + 2(j-1)Ag(D) + 2(j-1)Dg(A) - 2(j-1)g(E)\right) \\
&= AB^j + 2AB^{j-1}C + 2(j-1)A^2B^{j-2}D - 2(j-1)AB^{j-2}E + 2B^{j-1}D \\
&\quad - 2AB^{j-1}C - 2(j-1)AB^{j-2}E - 2(j-1)A^2B^{j-2}D + 4(j-1)B^{j-2}CD \\
&\quad + 2(j-1)B^{j-1}D + 4(j-1)AB^{j-2}E - 4(j-1)B^{j-2}CD \\
&= B^{j-1}\left(AB + 2D + 2(j-1)D\right) \\
&= B^{(j+1)-2}\left({\mathrm{div}_{\mathrm{H}}}^{(3)} \nabla_H\delta + 2((j+1)-2)D\right).
\end{aligned}
$$

Finally, using the result just obtained for ${\mathrm{div}_{\mathrm{H}}}^{(2j+1)} \nabla_H\delta$, we get the following

$$\begin{aligned}
\mathrm{div_H}^{(2(j+1))} \nabla_H \delta &= \mathrm{div_H}^{(2j+2)} \nabla_H \delta = A \ \mathrm{div_H}^{(2j+1)} \nabla_H \delta + g(\mathrm{div_H}^{(2j+1)} \nabla_H \delta) \\
&= AB^{j-1} \left(\mathrm{div_H}^{(3)} \nabla_H \delta + 2(j-1)D\right) + B^{j-1} g \left(\mathrm{div_H}^{(3)} \nabla_H \delta + 2(j-1)D\right) \\
&= AB^{j-1} \left(AB + 2D + 2(j-1)D\right) + B^{j-1} g \left(AB + 2D + 2(j-1)D\right) \\
&= AB^{j-1} \left(AB + 2jD\right) + B^{j-1} g \left(AB + 2jD\right) \\
&= A^2 B^j + 2jAB^{j-1}D + B^j g(A) + 2jB^{j-1} g(D) \\
&= A^2 B^j + 2jAB^{j-1}D - A^2 B^j + 2B^j C + B^{j+1} - 2B^{j-1} E \\
&= B^{j-1} \left(B^2 + 2BC + 2jAD - 2jE\right) \\
&= B^{j-1} \left(B^2 + 2BC + 2AD - 2E + 2(j-1)(AD - E)\right) \\
&= B^{(j+1)-2} \left(\mathrm{div_H}^{(4)} \nabla_H \delta + 2((j+1)-2)(AD - E)\right).
\end{aligned}$$

This completes the induction argument. □

4.2. **Analyticity of the volume function.** As an application of the recursive formula for the iterated divergences found in the previous section, we show that the volume function is analytic. This completes the proof of the main theorem of this paper.

Theorem 4.4. *For $\epsilon \geq 0$, let $\mathcal{T}_{\mathbb{H}}(B_0, \overline{\Omega}, \epsilon)$ be a localized Heisenberg ϵ-neighborhood of Ω as defined in Section 3.1. Then the function $a^{(0)} \colon [0, s_0] \to \mathbb{R}$ defined by*

$$a^{(0)}(\epsilon) = \mathcal{L}^3 \left(\mathcal{T}_{\mathbb{H}}(B_0, \overline{\Omega}, \epsilon)\right)$$

is real-analytic, and has a power series expansion at $\epsilon = 0$ given by

$$a^{(0)}(0) = \mathcal{L}^3(\overline{\Omega} \cap Q) + \sum_{i=1}^{\infty} \left(\int_{\partial\Omega \cap Q} (\mathrm{div_H}^{(i-1)} \nabla_H \delta) \, d\mathcal{H}^2_{\mathbb{R}^3} \right) \frac{\epsilon^i}{i!}.$$

Proof. We estimate $a^{(i)}(\epsilon)$ for a positive even integer $i = 2j$ and $\epsilon \in [0, s_0]$; as similar argument is valid when i is odd. By Theorem 3.5,

$$\begin{aligned}
\left|a^{(i)}(\epsilon)\right| &\leq \int_{\delta^{-1}(\epsilon) \cap Q} \left|\mathrm{div_H}^{(2j-1)} \nabla_H \delta\right| \, d\mathcal{H}^2_{\mathbb{R}^3} \\
&= \int_{\delta^{-1}(\epsilon) \cap Q} \left|AB^{j-1} + 2(j-1)B^{j-2}D\right| \, d\mathcal{H}^2_{\mathbb{R}^3} \\
&\leq \int_{\delta^{-1}(\epsilon) \cap Q} |A + B|^{j-1} + 2(j-1) \, |B + D|^{j-1} \, d\mathcal{H}^2_{\mathbb{R}^3}
\end{aligned}$$

The $\mathcal{C}^3$-smoothness of δ implies that the functions A, B, and D are $\mathcal{C}^1$-smooth on U. Combined with the above estimate and the continuity of the integral given by Lemma 3.6, this implies that there is a constant $M > 0$ satisfying

$$\sup_{\epsilon \in [0, s_0]} \left|a^{(i)}(\epsilon)\right| \leq M^{j-1}. \tag{4.17}$$

This and its counter-part for odd i quickly imply the analyticity of $a^{(0)}$. □

5. Examples

In this section we present two examples in which we can calculate the volume of a localized Heisenberg ϵ-tube both directly and with the aid of the series expansion proven in the previous sections. The basic idea in both examples is to view the localizing set Q as foliated by integral curves of the flow of the horizontal normal. This leads to a new coordinate system on the localized Heisenberg ϵ-tube that can be compared to the standard Euclidean coordinate system.

Example 5.1. We first consider the half-space

$$\mathbb{H}_{x_1^-} := \{(x_1, x_2, x_3) \in \mathbb{H} : x_1 < 0\},$$

with boundary

$$\partial\mathbb{H}_{x_1^-} = \{(x_1, x_2, x_3) \in \mathbb{H} : x_1 = 0\}.$$

This surface has no characteristic points, and the signed distance function is easy to compute, as follows. Let $g = (x_1, x_2, x_3) \in \mathbb{H}$, and consider the Cauchy problem

$$\begin{cases} \dot{\varphi}(s) = X_1(\varphi(s)), \\ \varphi(0) = (0, x_2, x_3 - 2x_2x_1). \end{cases}$$

Integrating, we see that

$$\varphi(s) = (s, x_2, x_3 - 2x_2(x_1 - s)),$$

and in particular $\varphi(x_1) = g$. Since the curve φ is horizontal and its projection to the plane $\{x_3 = 0\}$ is a straight line, φ is in fact a geodesic and

$$d_{CC}(\varphi(s), (0, x_2, x_3 - 2x_2(x_1 - s)) = s.$$

Using our standard notation, this implies that if $\Omega = \mathbb{H}_{x_1^-}$, then

$$\delta(x_1, x_2, x_3) = x_1, \text{ and } N = X_1.$$

Hence, given a Lipschitz domain $B_0 \subseteq \partial\mathbb{H}_{x_1^-}$ with Lipschitz boundary and any depth $s_0 > 0$, we see that the corresponding localizing set can be expressed by

$$Q = \{(s, x_2, x_3 + 2x_2s) : (0, x_2, x_3) \in B_0 \text{ and } |s| \leq s_0\},$$

and the the localized Heisenberg ϵ-tube can be expressed by

$$\mathcal{T}_{\mathbb{H}}(B_0, \overline{\mathbb{H}}_{x_1^-}, \epsilon) = \{(s, x_2, x_3 + 2x_2s) : (0, x_2, x_3) \in B_0 \text{ and } -s_0 \leq s \leq \epsilon\}.$$

The above coordinates allows us to compare $\mathcal{T}_{\mathbb{H}}(B_0, \overline{\mathbb{H}}_{x_1^-}, \epsilon)$ to the Euclidean tube

$$\mathcal{T}_{\mathbb{R}^3}(Q, \overline{\mathbb{H}}_{x_1^-}, \epsilon) = \{(s, x_2, x_3) : (0, x_2, x_3) \in B_0 \text{ and } -s_0 \leq s \leq \epsilon\},$$

which has volume $\mathcal{H}^2_{\mathbb{R}^3}(B_0)(s_0 + \epsilon)$. The mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3 + 2x_2x_1),$$

is a volume-preserving diffeomorphism. Since $f(\mathcal{T}_{\mathbb{R}^3}(B_0, \overline{\mathbb{H}}_{x_1^-}, \epsilon)) = \mathcal{T}_{\mathbb{H}}(B_0, \overline{\mathbb{H}}_{x_1^-}, \epsilon)$, the two tubes have equal volume. Noting that $||\nabla\delta||_{\mathbb{R}^3} = 1$, this fact along with the measure relation (3.8) shows that

$$\mathcal{L}^3(\mathcal{T}_{\mathbb{R}^3}\left(B_0, \overline{\mathbb{H}}_{x_1^-}, \epsilon)\right) = \mathcal{L}^3\left(\overline{\mathbb{H}}_{x_1^-} \cap Q\right) + \epsilon\left(\mathcal{H}^3_{d_{CC}}(B_0)\right).$$

Since δ has such a simple form, one can easily compute its iterated divergences and check the validity of our main theorem in this example.

Example 5.2. We now consider the half-space

$$\mathbb{H}_{x_3^-} = \{(x_1, x_2, x_3) \in \mathbb{H} : x_3 < 0\}$$

with boundary

$$\partial\mathbb{H}_{x_3^-} = \{(x_1, x_2, x_3) \in \mathbb{H} : x_3 = 0\},$$

which has a single characteristic point at $(0, 0, 0)$.

Unlike in the previous example, it is now non-trivial to calculate the corresponding signed distance function, the embedded horizontal normal, or its flow. However, general results from [1] give a formula for the integral curves of the embedded horizontal normal. The embedded horizontal normal itself can then be obtained by differentiation, and in certain cases (such as this example), a change of coordinates can be used to determine the signed distance function.

For $0 < r < R$, we consider the annulus

$$B_0 = \{(x_1, x_2, 0) \in \mathbb{R}^3 : r^2 < x_1^2 + x_2^2 < R\}$$

and denote by $U \subseteq \mathbb{H}$ an open set that contains the annulus B_0 and on which the signed-distance to $\partial\mathbb{H}_{x_3^-}$ function δ is $\mathcal{C}^3$. Denote by $N : U \to \mathbb{R}^3$ the embedded horizontal normal. By [1, Proposition 3.1], for $g = (g_1, g_2, 0) \in \partial\mathbb{H}_{x_3^-} \cap U$, the solution $\varphi_g = (\varphi_1, \varphi_2, \varphi_3)$ of the Cauchy problem

$$\begin{cases} \dot{\varphi}(s) = N(\varphi(s)), \\ \varphi(0) = g, \end{cases}$$

is given by

$$\varphi_g(s) = \begin{pmatrix} \varphi_1(s) \\ \varphi_2(s) \\ \varphi_3(s) \end{pmatrix} = \begin{pmatrix} \frac{g_1}{2}\left(1 + \cos\left(\frac{2s}{|g|}\right)\right) + \frac{g_2}{2}\sin\left(\frac{2s}{|g|}\right) \\ \frac{g_2}{2}\left(1 + \cos\left(\frac{2s}{|g|}\right)\right) - \frac{g_1}{2}\sin\left(\frac{2s}{|g|}\right) \\ \frac{|g|^2}{2}\left(\frac{2s}{|g|} + \sin\left(\frac{2s}{|g|}\right)\right). \end{pmatrix}, \tag{5.1}$$

where $|g| = (g_1^2 + g_2^2)^{1/2}$. Moreover, there is a number $s_0 > 0$ such that for any $g \in U_0$, the solution above exists on the interval $[-s_0, s_0]$.

We consider the localizing set Q generated by the set B_0 and depth s_0, i.e., we localize using the annulus and the flow defined above. We wish to calculate, for $0 < \epsilon < s_0$, the volume of the resulting tube $\mathcal{T}_{\mathbb{H}}(B_0, \overline{\mathbb{H}}_{x_3^-}, \epsilon)$.

We introduce a new coordinate system of Q, as in [2]. For each point $(x_1, x_2, x_3) \in Q$, we may find a $g \in B_0$ and $s \in [-s_0, s_0]$ so that

$$(x_1, x_2, x_3) = \varphi_g(s).$$

We first express g in polar coordinates as

$$g = (g_1, g_2, 0) = (\rho\cos\theta, \rho\sin\theta, 0),$$

and then set

$$\beta = \frac{s}{\rho} \text{ and } \alpha = \theta - \beta.$$

Our new coordinate system on Q is (ρ, α, β). Writing (5.1) in these coordinates and simplifying shows that

$$(x_1, x_2, x_3) = \varphi_g(s) = \left(\rho\cos(\beta)\cos(\alpha), \rho\cos(\beta)\sin(\alpha), \rho^2\left((\beta + \frac{\sin 2\beta}{2}\right)\right).$$

By Proposition 3.1,

$$\delta(x_1, x_2, x_3) = s = \rho\beta.$$

Hence, using the chain rule, we may calculate that

$$X_3\delta = \frac{1}{2\rho}, \quad (X_1\delta)(X_{32}\delta) - (X_2\delta)(X_{31}\delta) = -\frac{\cos\beta}{2\rho^2(\cos\beta + \beta\sin\beta)},$$

and

$$X_{33}\delta = -\frac{\sin\beta}{4\rho^3(\cos\beta + \beta\sin\beta)}, \quad (X_{31}\delta)^2 + (X_{32}\delta)^2 = \frac{\cos^2(\beta)}{4\rho^4(\cos\beta + \beta\sin\beta)^2}.$$

Setting $\beta = 0$ we can compute the functions A, B, C, D and E restricted to the plane $\partial\mathbb{H}_{x_3^-}$:

$$A = D = 0, \quad B = -\frac{4}{\rho^2}, \quad C = \frac{2}{\rho^2}, \quad E = \frac{4}{\rho^4}.$$

Note that the Euclidean outward pointing unit normal to $\mathbb{H}_{x_3^-}$ is $\nu = (0, 0, 1)$. Hence

$$||\nabla\delta||_{\mathbb{R}^3}^{-1} = \frac{||N||_{\mathrm{H}}}{||\nabla\delta||_{\mathbb{R}^3}^{-1}} = \sqrt{\langle X_1, \nu\rangle^2 + \langle X_2, \nu\rangle^2} = 2\sqrt{x_1^2 + x_2^2} = 2\rho.$$

It follows that

$$\int_{\partial\Omega\cap Q} (\mathrm{div_H}^{(0)}\, \nabla_H\delta)\ d\mathcal{H}_{CC}^3 = \int_{\partial\Omega\cap Q} ||\nabla\delta||_{\mathbb{R}^3}^{-1}\ d\mathcal{H}_{\mathbb{R}^3}^2$$
$$= \frac{4\pi}{3}(R^3 - r^3).$$

Moreover, for each integer $n \geq 0$, using (4.2) with $j = n + 1$ yields

$$\int_{\partial\Omega\cap Q} (\mathrm{div_H}^{(2n+2)}\, \nabla_H\delta)\, d\mathcal{H}_{CC}^3 = 2\int_0^{2\pi}\int_r^R \rho^2 \left(\left(\frac{-4}{\rho^2}\right)^{n-1}\left(\frac{-8n}{\rho^4}\right)\right) d\rho d\theta$$
$$= \frac{4\pi(-1)^n n 2^{2n+1}}{(1-2n)}(R^{1-2n} - r^{1-2n})$$

Plugging these results into the statement of Theorem 4.4 gives a Taylor series expansion for $\mathcal{L}^3(\mathcal{T}_{\mathbb{H}}(B_0, \overline{\mathbb{H}}_{x_3^-}, \epsilon))$ at $\epsilon = 0$.

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