The Definition of a Vector Space

Let V denote a nonempty set of objects, called *elements* or *vectors* on which we have defined two operations: *addition of vectors* and *multiplication of a vector by a scalar*. The set V is called a *vector space* if the following ten axioms are satisfied.

Closure axioms

1. CLOSURE UNDER ADDITION: $\forall\, \mathbf{x}, \mathbf{y} \in V\ \exists!$ element in V called the sum of $\mathbf{x}$ and $\mathbf{y}$, denoted by $\mathbf{x} + \mathbf{y}$.
2. CLOSURE UNDER MULTIPLICATION BY A REAL NUMBER: $\forall \mathbf{x} \in V$ and $\forall\, \alpha \in \mathbb{R}\ \exists$ an element in V called the product of α and $\mathbf{x}$, denoted by $\alpha\mathbf{x}$.

Axioms for addition

3. COMMUTATIVE LAW: $\forall\, \mathbf{x}, \mathbf{y} \in V,\ \ \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
4. ASSOCIATIVE LAW: $\forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in V,\ \ (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
5. EXISTENCE OF ZERO ELEMENT: $\exists$ an element of V, denoted by $\mathbf{0}$, $\ni\ \mathbf{x} + \mathbf{0} = \mathbf{x}$, $\forall\, \mathbf{x} \in V$.
6. EXISTENCE OF NEGATIVES: $\forall\, \mathbf{x} \in V$, the element (-1)$\mathbf{x}$ has the property that $\mathbf{x}$+(-1)$\mathbf{x}$=$\mathbf{0}$.

Axioms for multiplication by scalars

7. ASSOCIATIVE LAW: $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha,\ \beta \in \mathbb{R},\ \ \alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$.
8. DISTRIBUTIVE LAW FOR ADDITION IN V: $\forall\, \mathbf{x}, \mathbf{y} \in V$ and $\forall\, \alpha \in \mathbb{R}$, $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.
9. DISTRIBUTIVE LAW FOR ADDITION OF SCALARS: $\forall\, \mathbf{x} \in V$ and $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.
10. EXISTENCE OF IDENTITY: $\forall\, \mathbf{x} \in V$, 1$\mathbf{x}$=$\mathbf{x}$.

NOTES:

$\forall$ means 'for all' or 'for every.'
$\exists$ means 'there exists.'
$\exists!$ means 'there exists a unique.'
$\in$ means 'is an element of' or 'in.'
$\ni$ means 'such that.'
$\mathbb{R}$ represents the set of all real numbers. In this definition $\mathbb{R}$ may be replaced by $\mathbb{C}$, the set of all complex numbers or by any *field* $\mathbb{F}$.