The Fully Sinc-Galerkin Method for Time-Dependent Boundary Conditions

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The fully Sinc-Galerkin method is developed for a family of complex-valued partial differential equations with time-dependent boundary conditions. The Sinc-Galerkin discrete system is formulated and represented by a Kronecker product form of those equations. The numerical solution is efficiently calculated and the method exhibits an exponential convergence rate. Several examples, some with a real-valued solution and some with a complex-valued solution, are used to demonstrate the performance of this method. © 2004 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 20: 494–526, 2004

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I. INTRODUCTION

Fully Sinc-Galerkin techniques, ones that use a sinc function basis in both space and time, have become a powerful tool in the effort to efficiently solve time-dependent partial differential equations. The Sinc-Galerkin method for differential equations was originally introduced by Stenger [1, 2]. Since then, the Sinc-Galerkin method has been applied to a variety of ordinary [3, 4] and partial differential equations, including those with Dirichlet, Neumann, and mixed boundary conditions. The partial differential equations include Poisson's equation [5], the wave equation [6], the heat equation [7–9] the advection-diffusion equation [10], and beam equations [11]. The fully Sinc-Galerkin method was used successfully for time-dependent problems. Most importantly, this method exhibits an exponential order of convergence. Our goal is to develop the fully Sinc-Galerkin method for the efficient solution of a (possibly) complex-valued partial differential equation with time-dependent boundary conditions. To illustrate all the various possibilities we will focus on the family of complex-valued problems

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$$\mathcal{L}w(z, t) - ikw(z, t) = F(z, t), \qquad 0 < z < 1, 0 < t,$$

$$\mathcal{L}w(z, t) \equiv \frac{\partial w(z, t)}{\partial t} - A_v(t) \frac{\partial^2 w(z, t)}{\partial z^2}. \tag{1.1}$$

The function $A_v(t)$ could also depend on the variable z but in our applications it does not. The time-dependent boundary conditions and the initial condition are given by

$$\frac{\partial w(0, t)}{\partial z} = 0, \qquad 0 < t, \tag{1.2}$$

$$w(1, t) + \sigma A_v(t) \frac{\partial w(1, t)}{\partial z} = 0, \qquad 0 < t, \tag{1.3}$$

$$w(z, 0) = 0, \qquad 0 < z < 1. \tag{1.4}$$

Nonhomogeneous boundary conditions may be transformed to this homogeneous form via a simple change of dependent variable. Other combinations of boundary conditions can also be handled but this family will demonstrate all the necessary features.

II. SINC BASIS FUNCTIONS FOR THE GALERKIN METHOD

We will review sinc function properties, sinc quadrature rules, and the Sinc-Galerkin method. These are discussed thoroughly in [12] and [13]. To develop techniques for solving a partial differential equation, these properties will be used extensively in the following sections. The sinc function is defined for all $z \in \mathcal{C}$ by

$$\operatorname{sinc}(z) \equiv \begin{cases} \dfrac{\sin(\pi z)}{\pi z}, & \text{if } z \neq 0 \\ 1, & \text{if } z = 0. \end{cases}$$

In general, approximations can be constructed for infinite, semi-infinite, and finite intervals. Both spatial and temporal domains will be introduced. Define the function $w = \phi(z) = \ln[z/(1 - z)]$, which is a conformal mapping from $\mathcal{D}_E$, the eye-shaped spatial domain in the z-plane, onto the infinite strip, $\mathcal{D}_S$ in the w-plane, where

$$\mathcal{D}_E = \left\{ z = x + iy : \left| \arg\left(\frac{z}{1 - z}\right) \right| < d \leq \frac{\pi}{2} \right\},$$

$$\mathcal{D}_S = \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}.$$

This is shown in Fig. 1. For the Sinc-Galerkin method, the basis functions are derived from the composite translated sinc functions for $z \in \mathcal{D}_E$:

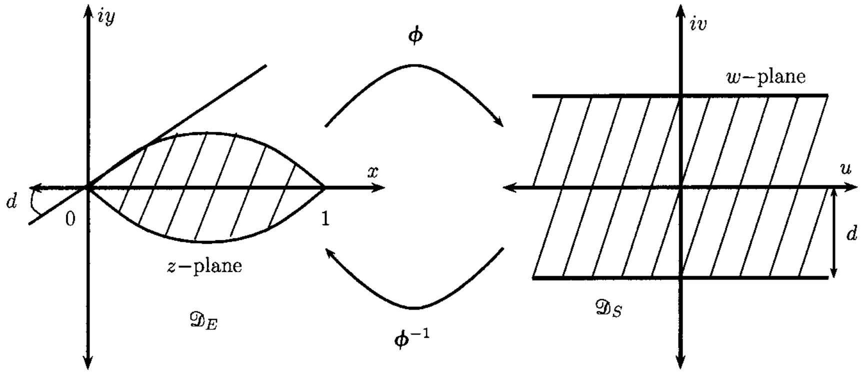

FIG. 1. Relationship between the domains $\mathscr{D}_E$ and $\mathscr{D}_S$.

$$S(k, h) \circ \phi(z) \equiv \operatorname{sinc}\left(\frac{\phi(z) - kh}{h}\right). \tag{2.1}$$

The sinc basis functions in (2.1) do not have a derivative when z tends to 0 or 1. To accommodate derivative boundary conditions, we modify the sinc basis functions as

$$\frac{S(k, h) \circ \phi(z)}{\phi'(z)} \equiv \frac{\operatorname{sinc}\left(\frac{\phi(z) - kh}{h}\right)}{\phi'(z)}, \tag{2.2}$$

where $1/[\phi'(z)] = z(1 - z)$. Note that the derivative of the modified sinc basis functions is defined as z approaches 0 or 1. These functions are shown in Fig. 2 for real values, x.

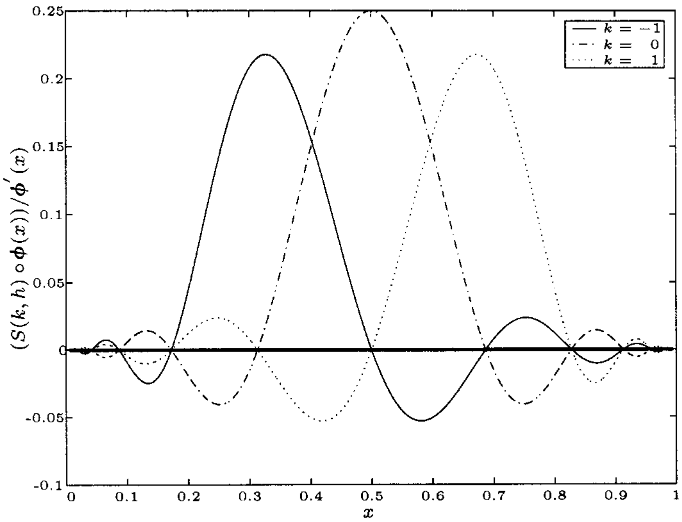

FIG. 2. Three adjacent members $[S(k, h) \circ \phi(x)]/[\phi'(x)]$, when $k = -1, 0, 1$ and $h = \pi/8$ of the modified sinc basis on the interval (0, 1).

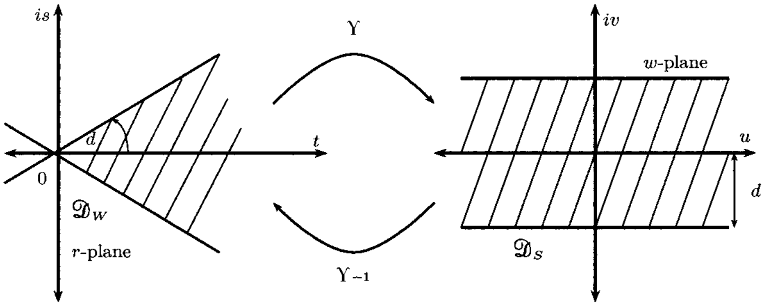

FIG. 3. Relationship between the domains $\mathcal{D}_W$ and $\mathcal{D}_S$.

For the temporal space, we construct an approximation by defining the function $w = \Upsilon(r) = \ln(r)$, which is a conformal mapping from $\mathcal{D}_W$, the wedge-shaped temporal domain, onto $\mathcal{D}_S$, the infinite strip, where

$$\mathcal{D}_W = \left\{ r = t + is : |\arg(r)| < d \leq \frac{\pi}{2} \right\}.$$

This is shown in Fig. 3. So the basis functions are derived from the composite translated sinc functions,

$$S(k, h) \circ \Upsilon(r) \equiv \operatorname{sinc}\left(\frac{\Upsilon(r) - kh}{h} \right), \tag{2.3}$$

for $r \in \mathcal{D}_W$. These functions are shown in Fig. 4 for real values, t.

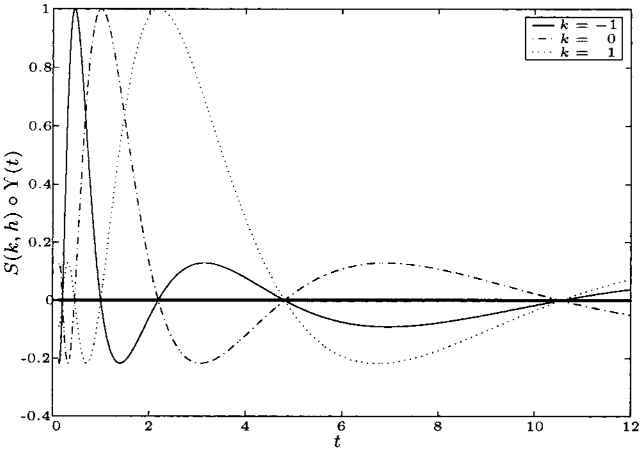

FIG. 4. Three adjacent members $S(k, h) \circ \Upsilon(t)$ when $k = -1, 0, 1$ and $h = \pi/8$ of the mapped sinc basis on $(0, \infty)$.

A. Sinc Quadrature Rules

The sinc quadrature rule (see [12, 13] for the proof) can provide the approximation of a double integral as follows.

Theorem 2.1. *For each fixed t, let $F(z, t) \in B(\mathcal{D}_E)$ and $h > 0$. Let ϕ and Υ be one-to-one conformal maps of the domains $\mathcal{D}_E$ and $\mathcal{D}_W$ onto $\mathcal{D}_S$, respectively. Let $z_j = \phi^{-1}(jh_z)$, $t_k = \Upsilon^{-1}(kh_t)$ and $\Gamma_z = \phi^{-1}(\mathbb{R})$, $\Gamma_t = \Upsilon^{-1}(\mathbb{R})$. Assume there are positive constants α_z, β_z, and $C_z(t)$ so that*

$$\left|\frac{F(z,t)}{\phi'(z)}\right| \le C_z(t)\begin{cases}\exp(-\alpha_z|\phi(z)|), & z \in \Gamma_a^{(z)}\\ \exp(-\beta_z|\phi(z)|), & z \in \Gamma_b^{(z)}\end{cases}$$

where $\Gamma_a^{(z)} \equiv \{z \in \Gamma_z : \phi(z) = u \in (-\infty, 0)\}$, $\Gamma_b^{(z)} \equiv \{z \in \Gamma_z : \phi(z) = u \in [0, \infty)\}$. Also for each fixed z, let $F(z, t) \in B(\mathcal{D}_W)$ and assume there are positive constants α_t, β_t, and $C_t(z)$ so that

$$\left|\frac{F(z,t)}{\Upsilon'(t)}\right| \le C_t(z)\begin{cases}\exp(-\alpha_t|\Upsilon(t)|), & t \in \Gamma_a^{(t)}\\ \exp(-\beta_t|\Upsilon(t)|), & t \in \Gamma_b^{(t)},\end{cases}$$

where

$$\Gamma_a^{(t)} \equiv \{t \in \Gamma_t : \Upsilon(t) = u \in (-\infty, 0)\}, \qquad \Gamma_b^{(t)} \equiv \{t \in \Gamma_t : \Upsilon(t) = u \in [0, \infty)\}.$$

Then the sinc trapezoidal quadrature rule is

$$\int_{\Gamma_t}\int_{\Gamma_z} F(z,t)\,dz\,dt = h_z h_t \sum_{j=-M_z}^{N_z}\sum_{k=-M_t}^{N_t} \frac{F(z_j, t_k)}{\phi'(z_j)\Upsilon'(t_k)} + \mathcal{O}(\exp(-\alpha_z M_z h_z)) + \mathcal{O}(\exp(-\beta_z N_z h_z))$$
$$+ \mathcal{O}(\exp(-2\pi d/h_z)) + \mathcal{O}(\exp(-\alpha_t M_t h_t)) + \mathcal{O}(\exp(-\beta_t N_t h_t)) + \mathcal{O}(\exp(-2\pi d/h_t)). \quad (2.4)$$

Hence, make the selections

$$N_z = \left[\left|\frac{\alpha_z}{\beta_z}M_z + 1\right|\right], \qquad M_t = \left[\left|\frac{\alpha_z}{\alpha_t}M_z + 1\right|\right], \qquad N_t = \left[\left|\frac{\alpha_z}{\beta_t}M_z + 1\right|\right],$$

where $h \equiv h_z = h_t$, and

$$h = \sqrt{2\pi d/(\alpha_z M_z)},$$

and the exponential order of the sinc trapezoidal quadrature rule is $\mathcal{O}(\exp(-\sqrt{2\pi d \alpha_z M_z}))$.

Corollary. *An important special case housed in the previous theorem occurs when the double integrand has the form $G(z, t)S(p, h_z) \circ \phi(z)S(q, h_t) \circ \Upsilon(t)$. Due to the interpolation*

$$S(p, h_z) \circ \phi(z_j) = S(p, h_z)(jh_z) = \delta_{jp}^{(0)} \qquad \text{and} \qquad S(q, h_t) \circ \Upsilon(t_k) = S(q, h_t)(kh_t) = \delta_{kq}^{(0)},$$

the sinc quadrature rule is a weighted point evaluation to the order of the method

$$\int_{\Gamma_t}\int_{\Gamma_z} G(z, t)S(p, h_z) \circ \phi(z)S(q, h_t) \circ \Upsilon(t)dzdt = h_z h_t \frac{G(z_p, t_q)}{\phi'(z_p)\Upsilon'(t_q)} + \mathbb{O}(\exp(-2\pi d/h_z)) + \mathbb{O}(\exp(-2\pi d/h_t)). \quad (2.5)$$

B. Matrix Representation of the Derivatives of Sinc Basis Functions at Nodal Points

The Sinc-Galerkin method actually requires the evaluated derivatives of sinc basis functions $S(p, h) \circ \phi(z)$ at the sinc nodes, $z = z_j$. The rth derivative of $S(p, h) \circ \phi(z)$, with respect to ϕ, evaluated at the nodal point z_j is denoted by

$$\frac{1}{h^r}\,\delta_{pj}^{(r)} \equiv \frac{d^r}{d\phi^r}\,[S(p, h) \circ \phi(z)]\big|_{z=z_j}. \quad (2.6)$$

The expressions in (2.6) for each p and j can be stored in a matrix $I^{(r)} = [\delta_{pj}^{(r)}]$.

For $r = 0, 1, 2,$

$$I^{(0)} = [\delta_{pj}^{(0)}], \qquad \text{where } \delta_{pj}^{(0)} \equiv [S(p, h) \circ \phi(z)]\big|_{z=z_j} = \begin{cases} 1, & \text{if } j = p \\ 0, & \text{if } j \neq p, \end{cases}$$

$$I^{(1)} = [\delta_{pj}^{(1)}], \qquad \text{where } \delta_{pj}^{(1)} \equiv h\,\frac{d}{d\phi}\,[S(p, h) \circ \phi(z)]\big|_{z=z_j} = \begin{cases} 0, & \text{if } j = p \\ \dfrac{(-1)^{j-p}}{j-p}, & \text{if } j \neq p, \end{cases}$$

$$I^{(2)} = [\delta_{pj}^{(2)}], \qquad \text{where } \delta_{pj}^{(2)} \equiv h^2\,\frac{d^2}{d\phi^2}\,[S(p, h) \circ \phi(z)]\big|_{z=z_j} = \begin{cases} \dfrac{-\pi^2}{3}, & \text{if } j = p \\ \dfrac{-2(-1)^{j-p}}{(j-p)^2}, & \text{if } j \neq p. \end{cases}$$

The following matrices are some examples for $I^{(0)}, I^{(1)}, I^{(2)}$. Given $-M_z - 1 \le p \le N_z + 1$ and $-M_z - 1 \le j \le N_z + 1$, $(m_z = M_z + N_z + 3)$, the $m_z \times m_z$, square matrices $I^{(0)}, I^{(1)}, I^{(2)}$ are given by

$$I^{(0)} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I,$$

$$I^{(1)} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & \cdots & \frac{(-1)^{m_z-1}}{m_z-1} \\ 1 & \ddots & \ddots & \ddots & \vdots \\ -\frac{1}{2} & \ddots & \ddots & \ddots & \frac{1}{2} \\ \vdots & \ddots & \ddots & \ddots & -1 \\ \frac{(-1)^{m_z}}{m_z-1} & \cdots & -\frac{1}{2} & 1 & 0 \end{bmatrix}, \quad I^{(2)} = \begin{bmatrix} -\frac{\pi^2}{3} & 2 & -\frac{2}{2^2} & \cdots & \frac{-2(-1)^{m_z-1}}{(m_z-1)^2} \\ 2 & \ddots & \ddots & \ddots & \vdots \\ -\frac{2}{2^2} & \ddots & \ddots & \ddots & -\frac{2}{2^2} \\ \vdots & \ddots & \ddots & \ddots & 2 \\ \frac{-2(-1)^{m_z-1}}{(m_z-1)^2} & \cdots & -\frac{2}{2^2} & 2 & -\frac{\pi^2}{3} \end{bmatrix}. \quad (2.7)$$

When $-M_z \le j \le N_z$, we remove the first and last columns of $I^{(0)}$, $I^{(1)}$, and $I^{(2)}$ in (2.7), to arrive at the $m_z \times n_z$, ($n_z = M_z + N_z + 1$), nonsquare matrices

$$I_z^{(0)} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix},$$

$$I_z^{(1)} = \begin{bmatrix} -1 & \frac{1}{2} & \cdots & \frac{(-1)^{m_z-2}}{m_z-2} \\ 0 & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \frac{1}{2} \\ -\frac{1}{2} & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & 0 \\ \frac{(-1)^{m_z-1}}{m_z-2} & \cdots & -\frac{1}{2} & 1 \end{bmatrix}, \quad I_z^{(2)} = \begin{bmatrix} 2 & -\frac{2}{2^2} & \cdots & \frac{-2(-1)^{m_z-2}}{(m_z-2)^2} \\ \frac{-\pi^2}{3} & \ddots & \ddots & \vdots \\ 2 & \ddots & \ddots & -\frac{2}{2^2} \\ -\frac{2}{2^2} & \ddots & \ddots & 2 \\ \vdots & \ddots & \ddots & \frac{-\pi^2}{3} \\ \frac{-2(-1)^{m_z-2}}{(m_z-2)^2} & \cdots & -\frac{2}{2^2} & 2 \end{bmatrix}. \tag{2.8}$$

When $-M_t \le q \le N_t + 1$ and $-M_t \le j \le N_t$, ($m_t = M_t + N_t + 2$ and $n_t = M_t + N_t + 1$), we remove the last column of $I^{(0)}$, $I^{(1)}$, and $I^{(2)}$ in (2.7), to leave the $m_t \times n_t$ nonsquare matrices

$$I_t^{(0)} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix},$$

$$I_t^{(1)} = \begin{bmatrix} 0 & -1 & \cdots & \frac{(-1)^{m_t-2}}{m_t-2} \\ 1 & \ddots & \ddots & \vdots \\ -\frac{1}{2} & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & 0 \\ \frac{(-1)^{m_t}}{m_t-1} & \cdots & -\frac{1}{2} & 1 \end{bmatrix}, \quad I_t^{(2)} = \begin{bmatrix} -\frac{\pi^2}{3} & 2 & \cdots & \frac{-2(-1)^{m_t-2}}{(m_t-2)^2} \\ 2 & \ddots & \ddots & \vdots \\ -\frac{2}{2^2} & \ddots & \ddots & 2 \\ \vdots & \ddots & \ddots & -\frac{\pi^2}{3} \\ \frac{-2(-1)^{m_t-1}}{(m_t-1)^2} & \cdots & -\frac{2}{2^2} & 2 \end{bmatrix}. \tag{2.9}$$

If a function f is evaluated at the sinc nodes $z = z_j$ for $-M_z \le j \le N_z$, then the $n_z \times n_z$ square diagonal matrix $\mathcal{D}_{n_z}(f)$ is written by

$$\mathcal{D}_{n_z}(f) = \begin{bmatrix} f(z_{-M_z}) & & & & 0 \\ & \ddots & & & \\ & & f(z_0) & & \\ & & & \ddots & \\ 0 & & & & f(z_{N_z}) \end{bmatrix}. \tag{2.10}$$

C. Matrices and Kronecker Products

We will develop the notation and tools to simplify discrete systems for complicated formulations. These tools are described in [12] and [14].

Definition. *For an* $m \times n$ *matrix* $\mathbf{B} = [b_{ij}]$, $1 \le i \le m$, $1 \le j \le n$,

$$\mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n], \qquad \text{where } \mathbf{b}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix},$$

the concatenation of $\mathbf{B}$ *is the* $mn \times 1$ *vector*

$$co(\mathbf{B}) = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}. \tag{2.11}$$

Definition. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix and* $\mathbf{B}$ *be a* $p \times q$ *matrix. The Kronecker or tensor product of* $\mathbf{A}$ *and* $\mathbf{B}$ *is the* $mp \times nq$ *matrix*

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix}. \tag{2.12}$$

A useful property of concatenation is given in the following theorem whose proof may be found in [14].

Theorem 2.2. *Let the linear system for the unknown matrix* $\mathbf{X}$ *be given as*

$$\mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{A}_2\mathbf{X}\mathbf{B}_2 + \cdots + \mathbf{A}_k\mathbf{X}\mathbf{B}_k = \mathbf{C}, \tag{2.13}$$

where $\mathbf{A}_i$ *are* $m \times m$, $\mathbf{X}$, $\mathbf{C}$ *are* $m \times n$, *and* $\mathbf{B}_i$ *are* $n \times n$. *This is equivalent to*

$$\mathbf{G}co(\mathbf{X}) = co(\mathbf{C}), \tag{2.14}$$

where $\mathbf{G} \equiv \mathbf{B}_1^T \otimes \mathbf{A}_1 + \mathbf{B}_2^T \otimes \mathbf{A}_2 + \cdots + \mathbf{B}_k^T \otimes \mathbf{A}_k$.

III. SOLVING THE PROBLEM WITH TIME-INDEPENDENT BOUNDARY CONDITIONS

The fully Sinc-Galerkin method is first applied to a simplified version of (1.1)–(1.4). This simplifies the initial formulation and will later allow for a convenient generalization to the time-dependent problem. The time-dependent boundary condition is first changed to a constant multiple σ and we choose $k = 0$ so that

$$\mathcal{L}w(z, t) \equiv \frac{\partial w(z, t)}{\partial t} - A_v(t)\frac{\partial^2 w(z, t)}{\partial z^2} = F(z, t), \qquad 0 < z < 1, \qquad 0 < t, \quad (3.1)$$

subject to the time-independent boundary conditions and the initial condition

$$\frac{\partial w(0, t)}{\partial z} = 0, \qquad 0 < t, \quad (3.2)$$

$$w(1, t) + \sigma\frac{\partial w(1, t)}{\partial z} = 0, \qquad 0 < t, \quad (3.3)$$

$$w(z, 0) = 0, \qquad 0 < z < 1. \quad (3.4)$$

The Sinc-Galerkin procedure begins by selecting composite sinc functions appropriate to each domain (0, 1) and (0, ∞) as the sinc basis functions for the approximate expansion of the solution $w(z, t)$. The conformal mapping functions on the spatial domain (0, 1) and the temporal domain (0, ∞) are given by

$$\phi(z) = \ln\left(\frac{z}{1-z}\right), \qquad \Upsilon(t) = \ln(t),$$

respectively, and the appropriate composite sinc functions $S(p, h_z) \circ \phi$ over (0, 1) and $S(q, h_t) \circ \Upsilon$ over (0, ∞) are given by

$$S(p, h_z) \circ \phi(z) \equiv \operatorname{sinc}\left(\frac{\phi(z) - ph_z}{h_z}\right) \equiv \begin{cases} \dfrac{\sin\left(\dfrac{\pi}{h_z}(\phi(z) - ph_z)\right)}{\dfrac{\pi}{h_z}(\phi(z) - ph_z)}, & \phi(z) \neq ph_z \\ 1, & \phi(z) = ph_z, \end{cases}$$

$$S(q, h_t) \circ \Upsilon(t) \equiv \operatorname{sinc}\left(\frac{\Upsilon(t) - qh_t}{h_t}\right) \equiv \begin{cases} \dfrac{\sin\left(\dfrac{\pi}{h_t}(\Upsilon(t) - qh_t)\right)}{\dfrac{\pi}{h_t}(\Upsilon(t) - qh_t)}, & \Upsilon(t) \neq qh_t \\ 1, & \Upsilon(t) = qh_t. \end{cases}$$

Consider the set of sinc basis functions for the one-dimensional transient problem (3.1)–(3.4) as the product sinc basis functions ($-M_z - 1 \leq p \leq N_z + 1$ and $-M_t \leq q \leq N_t + 1$)

$$\{S_p(z)S_q^*(t) \equiv (S(p, h_z) \circ \phi(z))(S(q, h_t) \circ \Upsilon(t))\}. \tag{3.5}$$

For the boundary conditions in (3.2) and (3.3), the sinc basis functions (3.5) tend to zero as z approaches to 0 and 1 but their z partial derivatives are undefined at $z = 0$ and $z = 1$. To remedy this it is necessary to both modify the sinc basis functions as shown in (2.2) and add some additional boundary basis functions. Following [12], Hermite functions, whose derivatives are defined at the two endpoints, are selected to be boundary basis functions for the spatial domain as

$$w_0(z) = (2z + 1)(1 - z)^2, \qquad w_1(z) = (1 - z)z^2 + \sigma(3 - 2z)z^2. \tag{3.6}$$

Also the sinc basis functions (3.5) tend to zero as t approaches ∞, so a nonzero bounded function is chosen to be an additional basis function for the temporal space

$$w_\infty(t) = \frac{t}{1 + t}. \tag{3.7}$$

The approximate solution for $w(z, t)$ is represented by the expansion, ($m_z = M_z + N_z + 3$, $n_z = M_z + N_z + 1$, $m_t = M_t + N_t + 2$, $n_t = M_t + N_t + 1$),

$$w_a(z, t) = \sum_{j=-M_z-1}^{N_z+1} \sum_{k=-M_t}^{N_t+1} c_{jk}\xi_j(z)\zeta_k(t), \tag{3.8}$$

where

$$\xi_j(z) = \begin{cases} w_0(z), & \text{if } j = -M_z - 1 \\ \left(\dfrac{S_j}{\phi'}\right)(z), & \text{if } j = -M_z, \ldots, N_z, \\ w_1(z), & \text{if } j = N_z + 1, \end{cases} \qquad \zeta_k(t) = \begin{cases} S_k^*(t), & \text{if } k = -M_t, \ldots, N_t \\ w_\infty(t), & \text{if } k = N_t + 1. \end{cases} \tag{3.9}$$

Approaching the formulation of the fully Sinc-Galerkin method, we start by rewriting the approximate solution $w_a(z, t)$ in (3.8) as

$$w_a(z, t) = w_{a_1}(z, t) + w_{a_2}(z, t) + w_{a_3}(z, t) + w_{a_4}(z, t) + w_{a_5}(z, t) + w_{a_6}(z, t),$$

where

$$w_{a_1}(z, t) = \sum_{k=-M_t}^{N_t} c_{(-M_z-1)k}w_0(z)S_k^*(t), \qquad w_{a_2}(z, t) = c_{(-M_z-1)(N_t+1)}w_0(z)w_\infty(t),$$

$$w_{a_3}(z,t) = \sum_{j=-M_z}^{N_z} \sum_{k=-M_t}^{N_t} c_{jk}\left(\frac{S_j}{\phi'}\right)(z) S_k^*(t), \qquad w_{a_4}(z,t) = \sum_{j=-M_z}^{N_z} c_{j(N_t+1)}\left(\frac{S_j}{\phi'}\right)(z) w_\infty(t),$$

$$w_{a_5}(z,t) = \sum_{k=-M_t}^{N_t} c_{(M_z+1)k} w_1(z) S_k^*(t), \qquad w_{a_6}(z,t) = c_{(M_z+1)(N_t+1)} w_1(z) w_\infty(t). \quad (3.10)$$

The unknown coefficients c_{jk} are determined by orthogonalizing the residual with respect to the sinc basis functions in (3.5). This leads to the equations ($-M_z - 1 \le p \le N_z + 1$, $-M_t \le q \le N_t + 1$)

$$(\mathcal{L}w_{a_1}, S_pS_q^*) + (\mathcal{L}w_{a_2}, S_pS_q^*) + (\mathcal{L}w_{a_3}, S_pS_q^*) + (\mathcal{L}w_{a_4}, S_pS_q^*) + (\mathcal{L}w_{a_5}, S_pS_q^*) + (\mathcal{L}w_{a_6}, S_pS_q^*) = (F, S_pS_q^*), \quad (3.11)$$

where the inner product and the choice of weight function are given by

$$(f, g) = \int_0^\infty \int_0^1 f(z,t) g(z,t) \varpi(z,t)\, dz dt,$$

$$\varpi(z,t) = \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}.$$

To each inner product in (3.11) we apply the sinc quadrature rule. This leads to the construction of the discrete system that approximates (3.11). From (3.11), the inner product involving $\mathcal{L}w_{a_1}$ is

$$(\mathcal{L}w_{a_1}, S_pS_q^*) = \sum_{k=-M_t}^{N_t} c_{(-M_z-1)k}[(w_0(S_k^*)', S_pS_q^*) - (A_v w_0'' S_k^*, S_pS_q^*)]. \quad (3.12)$$

The first inner product in (3.12) is evaluated by first integrating by parts once in the variable t and then applying the sinc quadrature rule in space and time. First we get

$$(w_0(S_k^*)', S_pS_q^*) = \int_0^\infty \int_0^1 w_0(z)(S_k^*(t))' S_p(z) S_q^*(t) \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dz dt$$
$$= \int_0^1 \frac{w_0(z) S_p(z)}{\sqrt{\phi'(z)}} \left[B_{T_1} - \int_0^\infty S_k^*(t)(S_q^* \sqrt{\Upsilon'})'(t) dt \right] dz. \quad (3.13)$$

From [12], the boundary condition term

$$B_{T_1} = (S_k^* S_q^* \sqrt{\Upsilon'})(t)\big|_0^\infty \quad (3.14)$$

equals zero, and

$$(S_q^* \sqrt{\Upsilon'})'(t) = \left(\frac{dS_q^*(t)}{d\Upsilon} - \frac{1}{2} S_q^*(t) \right) (\Upsilon'(t))^{3/2}.$$

Then applying the sinc quadrature rule (2.5), (3.13) becomes

$$\begin{aligned}(w_0(S_k^*)', S_pS_q^*) &= \int_0^1 \frac{w_0(z)S_p(z)}{\sqrt{\phi'(z)}}\, dz \int_0^\infty S_k^*(t)\left(-\frac{dS_q^*(t)}{d\Upsilon} + \frac{1}{2} S_q^*(t)\right)(\Upsilon'(t))^{3/2}dt \\ &\approx h_z h_t \frac{w_0(z_p)}{(\phi'(z_p))^{3/2}} \left\{ -\frac{1}{h_t} \delta_{qk}^{(1)} + \frac{1}{2} \delta_{qk}^{(0)} \right\} \sqrt{\Upsilon'(t_k)}. \qquad (3.15)\end{aligned}$$

The last inner product in (3.12) is handled by the direct application of the sinc quadrature rule (2.5)

$$(A_v w_0'' S_k^*, S_pS_q^*) = \int_0^\infty \int_0^1 w_0''(z)A_v(t)S_k^*(t)S_p(z)S_q^*(t) \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dzdt \approx h_z h_t \frac{w_0''(z_p)}{(\phi'(z_p))^{3/2}} \frac{A_v(t_k)}{\sqrt{\Upsilon'(t_k)}}. \qquad (3.16)$$

So

$$\begin{aligned}(\mathcal{L}w_{a_1}, S_pS_q^*) \approx h_z h_t \sum_{k=-M_t}^{N_t} \Bigg[&\frac{w_0(z_p)}{(\phi'(z_p))^{3/2}}\, c_{(-M_z-1)k} \left\{ -\frac{1}{h_t} \delta_{qk}^{(1)} + \frac{1}{2} \delta_{qk}^{(0)} \right\} \sqrt{\Upsilon'(t_k)} \\ &- \frac{w_0''(z_p)}{(\phi'(z_p))^{3/2}}\, c_{(-M_z-1)k} \frac{A_v(t_k)}{\sqrt{\Upsilon'(t_k)}} \Bigg].\end{aligned}$$

Letting p, q range over all values gives the matrix approximation to $(\mathcal{L}w_{a_1}, S_pS_q^*)$,

$$h_z h_t \{\mathcal{A}_1 \mathcal{C} \mathcal{D}_{t_1} + \mathcal{D}_{s_1} \mathcal{C} \mathcal{B}_1\}, \qquad (3.17)$$

with the $m_z \times m_z$ ($m_z = M_z + N_z + 3$) block matrices

$$\begin{aligned}\mathcal{A}_1 &\equiv [\mathbf{a}_{-M_z-1}^0 | \mathbf{O}_{m_z \times n_z} | \mathbf{O}_{m_z \times 1}], \\ \mathcal{D}_{s_1} &\equiv [\mathbf{b}_{-M_z-1}^0 | \mathbf{O}_{m_z \times n_z} | \mathbf{O}_{m_z \times 1}], \qquad (3.18)\end{aligned}$$

the $m_t \times m_t$ ($m_t = M_t + N_t + 2$) block matrices

$$\mathcal{D}_{t_1} \equiv \left[\begin{array}{c|c} \mathcal{D}_{n_t}\left(\frac{1}{\sqrt{\Upsilon'}} \right) & \mathbf{O}_{n_t \times 1} \\ \hline \multicolumn{2}{c}{\mathbf{O}_{1 \times m_t}} \end{array} \right] \mathcal{D}_{m_t}(A_v), \qquad \mathcal{B}_1 \equiv \left[\frac{\mathbf{B}_w^T}{\mathbf{O}_{1 \times m_t}} \right], \qquad (3.19)$$

and the nonsquare $m_t \times n_t$ ($n_t = M_t + N_t + 1$) matrix

$$\mathbf{B}_w \equiv \left\{ -\frac{1}{h_t} I_t^{(1)} + \frac{1}{2} I_t^{(0)} \right\} \mathcal{D}_{n_t}(\sqrt{\Upsilon'}). \tag{3.20}$$

Here the $m_t \times n_t$ matrices $I_t^{(0)}$, $I_t^{(1)}$ are from (2.9), and the $m_z \times 1$ column vectors $\mathbf{a}^0_{-M_z-1}$ and $\mathbf{b}^0_{-M_z-1}$ have pth component, $p = -M_z - 1, -M_z, \ldots, N_z, N_z + 1$,

$$[\mathbf{a}^0_{-M_z-1}]_p \equiv -\frac{w_0''(z_p)}{(\phi'(z_p))^{3/2}}, \qquad [\mathbf{b}^0_{-M_z-1}]_p \equiv \frac{w_0(z_p)}{(\phi'(z_p))^{3/2}}. \tag{3.21}$$

The diagonal matrices

$$\mathcal{D}_{n_t}\left(\frac{1}{\sqrt{\Upsilon'}}\right) \equiv \begin{bmatrix} \dfrac{1}{\sqrt{\Upsilon'(t_{-M_t})}} & & \\ & \ddots & \\ & & \dfrac{1}{\sqrt{\Upsilon'(t_{N_t})}} \end{bmatrix},$$

$$\mathcal{D}_{m_t}(A_v) \equiv \begin{bmatrix} A_v(t_{-M_t}) & & \\ & \ddots & \\ & & A_v(t_{N_t+1}) \end{bmatrix}. \tag{3.22}$$

Next, the inner product $(\mathcal{L}w_{a_5}, S_p S_q^*)$ in (3.11) is similar to $(\mathcal{L}w_{a_1}, S_p S_q^*)$, so we get

$$(\mathcal{L}w_{a_5}, S_p S_q^*) \approx h_z h_t \sum_{k=-M_t}^{N_t} \left[\frac{w_1(z_p)}{(\phi'(z_p))^{3/2}} c_{(M_z+1)k} \left\{ -\frac{1}{h_t} \delta_{qk}^{(1)} + \frac{1}{2} \delta_{qk}^{(0)} \right\} \sqrt{\Upsilon'(t_k)} - \frac{w_1''(z_p)}{(\phi'(z_p))^{3/2}} c_{(M_z+1)k} \frac{A_v(t_k)}{\sqrt{\Upsilon'(t_k)}} \right].$$

Letting p, q range over all values gives the matrix approximation to $(\mathcal{L}w_{a_5}, S_p S_q^*)$,

$$h_z h_t \{ \mathcal{A}_5 \mathcal{C} \mathcal{D}_{t_1} + \mathcal{D}_{s_5} \mathcal{C} \mathcal{B}_1 \}, \tag{3.23}$$

where $\mathcal{D}_{t_1}$, $\mathcal{B}_1$ are in (3.19), the $m_z \times m_z$ block matrices

$$\mathcal{A}_5 \equiv [\mathbf{O}_{m_z \times 1} | \mathbf{O}_{m_z \times n_z} | \mathbf{a}^1_{N_z+1}],$$

$$\mathcal{D}_{s_5} \equiv [\mathbf{O}_{m_z \times 1} | \mathbf{O}_{m_z \times n_z} | \mathbf{b}^1_{N_z+1}], \tag{3.24}$$

and the $m_z \times 1$ column vectors $\mathbf{a}^1_{N_z+1}$ and $\mathbf{b}^1_{N_z+1}$ have pth component, $p = -M_z - 1, -M_z, \ldots, N_z, N_z + 1$,

$$[\mathbf{a}^1_{N_z+1}]_p \equiv -\frac{w''_1(z_p)}{(\phi'(z_p))^{3/2}}, \qquad [\mathbf{b}^1_{N_z+1}]_p \equiv \frac{w_1(z_p)}{(\phi'(z_p))^{3/2}}. \tag{3.25}$$

Since $(\mathcal{L}w_{a_2}, S_pS_q^*) = c_{(-M_z-1)(N_t+1)}((w_0 w'_\infty, S_pS_q^*) - (A_v w''_0 w_\infty, S_pS_q^*))$ in (3.11), we must approximate

$$(w_0 w'_\infty, S_pS_q^*) = \int_0^\infty \int_0^1 w_0(z) w'_\infty(t) S_p(z) S_q^*(t) \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dz dt,$$

$$(A_v w''_0 w_\infty, S_pS_q^*) = \int_0^\infty \int_0^1 w''_0(z) w_\infty(t) A_v(t) S_p(z) S_q^*(t) \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dz dt. \tag{3.26}$$

Applying the sinc quadrature rule (2.5), the result is

$$(\mathcal{L}w_{a_2}, S_pS_q^*) \approx h_z h_t \frac{w_0(z_p)}{(\phi'(z_p))^{3/2}} c_{(-M_z-1)(N_t+1)} \frac{w'_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}} - h_z h_t \frac{w''_0(z_p)}{(\phi'(z_p))^{3/2}} c_{(-M_z-1)(N_t+1)} A_v(t_q) \frac{w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}.$$

Letting p, q range over all values gives the matrix approximation to $(\mathcal{L}w_{a_2}, S_pS_q^*)$,

$$h_z h_t \{\mathcal{A}_1 \mathcal{C} \mathcal{D}_{t_2} + \mathcal{D}_{s_1} \mathcal{C} \mathcal{B}_2\}, \tag{3.27}$$

where $\mathcal{A}_1$, $\mathcal{D}_{s_1}$ are in (3.18), the $m_t \times m_t$ block matrix

$$\mathcal{B}_2 \equiv \left[\frac{\mathbf{O}_{n_t \times m_t}}{\mathbf{a}_\infty^T}\right], \qquad \mathcal{D}_{t_2} \equiv \left[\frac{\mathbf{O}_{n_t \times n_t} | \mathbf{O}_{n_t \times 1}}{\mathbf{b}_\infty^T}\right] \mathcal{D}_{m_t}(A_v), \tag{3.28}$$

and the $m_t \times 1$ column vectors $\mathbf{a}_\infty$ and $\mathbf{b}_\infty$ have qth component, $q = -M_t, -M_t + 1, \ldots, N_t, N_t + 1$,

$$[\mathbf{a}_\infty]_q \equiv \frac{w'_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}, \qquad [\mathbf{b}_\infty]_q \equiv \frac{w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}. \tag{3.29}$$

From (3.11), w_{a_6} is similar to w_{a_2}, so the inner product is approximated by

$$(\mathcal{L}w_{a_6}, S_pS_q^*) \approx h_z h_t \left(\frac{w_1(z_p)}{(\phi'(z_p))^{3/2}}\right) c_{(M_z+1)(N_t+1)} \left(\frac{w'_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}\right) - h_z h_t \left(\frac{w''_1(z_p)}{(\phi'(z_p))^{3/2}}\right) c_{(M_z+1)(N_t+1)} A_v(t_q) \left(\frac{w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}\right).$$

Letting p, q range over all values gives the matrix approximation to $(\mathcal{L}w_{a_6}, S_pS_q^*)$,

$$h_z h_t\{\mathcal{A}_5 \mathcal{C} \mathcal{D}_{t_2} + \mathcal{D}_{s_5} \mathcal{C} \mathcal{B}_2\}, \tag{3.30}$$

where $\mathcal{D}_{t_2}$ and $\mathcal{B}_2$ are in (3.28) and $\mathcal{A}_5$ and $\mathcal{D}_{s_5}$ are in (3.24). From (3.11), the inner product

$$(\mathcal{L} w_{a_3}, S_p S_q^*) = \sum_{j=-M_z}^{N_z} \sum_{k=-M_t}^{N_t} c_{jk}\left(\left(\frac{S_j}{\phi'}\right)(S_k^*)', S_p S_q^*\right) - \sum_{j=-M_z}^{N_z} \sum_{k=-M_t}^{N_t} c_{jk}\left(A_v\left(\frac{S_j}{\phi'}\right)'' S_k^*, S_p S_q^*\right). \tag{3.31}$$

Applying integration by parts in t to the first inner product in (3.31), we get

$$\begin{aligned}\left(\left(\frac{S_j}{\phi'}\right)(S_k^*)', S_p S_q^*\right) &= \int_0^\infty \int_0^1 \frac{S_j(z)}{\phi'(z)} (S_k^*(t))' S_p(z) S_q^*(t) \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dz dt \\ &= \int_0^1 \frac{S_j(z) S_p(z)}{(\phi'(z))^{3/2}} \left[B_{T_1} - \int_0^\infty S_k^*(t)(S_q^* \sqrt{\Upsilon'})'(t) dt \right] dz.\end{aligned}$$

The boundary condition term B_{T_1} in (3.14) equals zero, so we get

$$\begin{aligned}\left(\left(\frac{S_j}{\phi'}\right)(S_k^*)', S_p S_q^*\right) &= \int_0^1 \frac{S_j(z) S_p(z)}{(\phi'(z))^{3/2}}\, dz \int_0^\infty S_k^*(t)\left(-\frac{dS_q^*(t)}{d\Upsilon} + \frac{1}{2} S_q^*(t)\right)(\Upsilon'(t))^{3/2} dt \\ &\approx h_z h_t \frac{1}{(\phi'(z_j))^{5/2}} \left\{-\frac{1}{h_t} \delta_{qk}^{(1)} + \frac{1}{2} \delta_{qk}^{(0)}\right\} \sqrt{\Upsilon'(t_k)}\,. \end{aligned} \tag{3.32}$$

The second inner product in (3.31) is

$$\begin{aligned}\left(A_v\left(\frac{S_j}{\phi'}\right)'' S_k^*, S_p S_q^*\right) &= \int_0^\infty \int_0^1 A_v(t)\left(\frac{S_j(z)}{\phi'(z)}\right)'' S_k^*(t) S_p(z) S_q^*(t) \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dz dt \\ &= \int_0^\infty (A_v S_k^* S_q^* \sqrt{\Upsilon'})(t) \left[B_{T_2} + \int_0^1 \frac{S_j(z)}{\phi'(z)} \left(\frac{S_p(z)}{\sqrt{\phi'(z)}}\right)'' dz \right] dt, \end{aligned} \tag{3.33}$$

where

$$B_{T_2} = \left[\left(\frac{S_j(z)}{\phi'(z)}\right)'\left(\frac{S_p(z)}{\sqrt{\phi'(z)}}\right) - \left(\frac{S_j(z)}{\phi'(z)}\right)\left(\frac{S_p(z)}{\sqrt{\phi'(z)}}\right)'\right]\Bigg|_0^1 \tag{3.34}$$

equals zero, and

$$\left(\frac{S_p(z)}{\sqrt{\phi'(z)}}\right)'' = \left(\frac{d^2 S_p(z)}{d\phi^2} - \frac{1}{4} S_p(z)\right)(\phi'(z))^{3/2}.$$

So (3.33) becomes

$$\left(A_v\left(\frac{S_j}{\phi'}\right)'' S_k^*, S_pS_q^*\right) \approx h_zh_t\left\{\frac{1}{h_z^2}\,\delta_{pj}^{(2)} - \frac{1}{4}\,\delta_{pj}^{(0)}\right\}\frac{1}{\sqrt{\phi'(z_j)}}\left(\frac{A_v(t_k)}{\sqrt{\Upsilon'(t_k)}}\right).$$

Then the inner product (3.31) is

$$(\mathscr{L}w_{a_3}, S_pS_q^*) \approx \sum_{j=-M_z}^{N_z}\sum_{k=-M_t}^{N_t} h_zh_t\,\frac{c_{jk}}{(\phi'(z_j))^{5/2}}\left\{-\frac{1}{h_t}\,\delta_{qk}^{(1)} + \frac{1}{2}\,\delta_{qk}^{(0)}\right\}\sqrt{\Upsilon'(t_k)}$$
$$+\sum_{j=-M_z}^{N_z}\sum_{k=-M_t}^{N_t} h_zh_t\left\{-\frac{1}{h_z^2}\,\delta_{pj}^{(2)} + \frac{1}{4}\,\delta_{pj}^{(0)}\right\}\frac{c_{jk}}{\sqrt{\phi'(z_j)}}\,\frac{A_v(t_k)}{\sqrt{\Upsilon'(t_k)}}.$$

Letting p, q range over all values gives the matrix approximation to $(\mathscr{L}w_{a_3}, S_pS_q^*)$,

$$h_zh_t\{\mathscr{A}_3\mathscr{C}\mathscr{D}_{t_1} + \mathscr{D}_{s_3}\mathscr{C}\mathscr{B}_1\}, \tag{3.35}$$

where $\mathscr{D}_{t_1}$ and $\mathscr{B}_1$ are defined in (3.19), the $m_z \times m_z$ block matrices

$$\mathscr{A}_3 \equiv [\mathbf{O}_{m_z\times 1}|\mathbf{A}_w|\mathbf{O}_{m_z\times 1}],$$

$$\mathscr{D}_{s3} \equiv \left[\mathbf{O}_{m_z\times 1}\left|I_z^{(0)}\mathscr{D}_{n_z}\left(\frac{1}{(\phi')^{5/2}}\right)\right|\mathbf{O}_{m_z\times 1}\right], \tag{3.36}$$

and the nonsquare $m_z \times n_z$ matrix

$$\mathbf{A}_w \equiv \left\{-\frac{1}{h_z^2}\,I_z^{(2)} + \frac{1}{4}\,I_z^{(0)}\right\}\mathscr{D}_{n_z}\left(\frac{1}{\sqrt{\phi'}}\right). \tag{3.37}$$

Lastly, we have $(\mathscr{L}w_{a_4}, S_pS_q^*)$ in (3.11) such that

$$(\mathscr{L}w_{a_4}, S_pS_q^*) = \sum_{j=-M_z}^{N_z} c_{j(N_t+1)}\left(\left(\frac{S_j}{\phi'}\right)w'_\infty, S_pS_q^*\right) - \sum_{j=-M_z}^{N_z} c_{j(N_t+1)}\left(A_v(t)\left(\frac{S_j}{\phi'}\right)'' w_\infty, S_pS_q^*\right). \tag{3.38}$$

Applying the sinc quadrature rule (2.5) to the first inner product in (3.38), the result is

$$\left(\left(\frac{S_j}{\phi'}\right)w'_\infty, S_pS_q^*\right) = \int_0^\infty\int_0^1 \frac{S_j(z)}{\phi'(z)}\,w'_\infty(t)S_p(z)S_q^*(t)\sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\,dz\,dt \approx h_zh_t\,\frac{1}{(\phi'(z_j))^{5/2}}\,\frac{w'_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}.$$

Next, the second inner product in (3.38) is integrated by parts twice in z

$$\left(A_v\left(\frac{S_j}{\phi'}\right)'' w_\infty,\ S_pS_q^*\right) = \int_0^\infty \int_0^1 A_v(t)\left(\frac{S_j(z)}{\phi'(z)}\right)'' w_\infty(t)S_p(z)S_q^*(t)\sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\,dzdt$$
$$= \int_0^\infty (A_v w_\infty S_q^*\sqrt{\Upsilon'})(t)\left[B_{T_2} + \int_0^1 \frac{S_j(z)}{\phi'(z)}\left(\frac{S_p(z)}{\sqrt{\phi'(z)}}\right)'' dz\right]dt,$$

where the boundary condition term B_{T_2} given in (3.34) equals zero. The result is

$$\left(A_v\left(\frac{S_j}{\phi'}\right)'' w_\infty,\ S_pS_q^*\right) \approx h_zh_t\left\{\frac{1}{h_z^2}\,\delta_{pj}^{(2)} - \frac{1}{4}\,\delta_{pj}^{(0)}\right\}\frac{1}{\sqrt{\phi'(z_j)}}\,\frac{A_v(t_q)w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}.$$

Then the inner product (3.38) is

$$(\mathcal{L}w_{a_4},\ S_pS_q^*) \approx h_zh_t\sum_{j=-M_z}^{N_z}\frac{1}{(\phi'(z_j))^{5/2}}\,c_{j(N_t+1)}\,\frac{w_\infty'(t_q)}{\sqrt{\Upsilon'(t_q)}}$$
$$+\ h_zh_t\sum_{j=-M_z}^{N_z}\left\{-\frac{1}{h_z^2}\,\delta_{pj}^{(2)} + \frac{1}{4}\,\delta_{pj}^{(0)}\right\}\frac{1}{\sqrt{\phi'(z_j)}}\,c_{j(N_t+1)}\,\frac{A_v(t_q)w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}.$$

Letting p, q range over all values gives the matrix approximation to $(\mathcal{L}w_{a_4},\ S_pS_q^*)$,

$$h_zh_t\{\mathcal{A}_3\mathcal{C}\mathcal{D}_{t_2} + \mathcal{D}_{s_3}\mathcal{C}\mathcal{B}_2\}, \tag{3.39}$$

where $\mathcal{A}_3$, $\mathcal{D}_{s_3}$ are in (3.36) and $\mathcal{D}_{t_2}$, $\mathcal{B}_2$ are in (3.28).

Applying the sinc quadrature rule (2.5), the inner product in the right-hand side of (3.11) leads to

$$(F,\ S_pS_q^*) = \int_0^\infty \int_0^1 F(z,\ t)S_p(z)S_q^*(t)\sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\,dzdt \approx h_zh_t\,\frac{1}{(\phi'(z_p))^{3/2}}\,F(z_p,\ t_q)\,\frac{1}{\sqrt{\Upsilon'(t_q)}}. \tag{3.40}$$

Letting p, q range over all values gives the matrix approximation to $(F,\ S_pS_q^*)$,

$$h_zh_t\mathcal{D}_{m_z}\left(\frac{1}{(\phi')^{3/2}}\right)\mathbf{F}\mathcal{D}_{m_t}\left(\frac{1}{\sqrt{\Upsilon'}}\right), \tag{3.41}$$

where the pqth-entry of $\mathbf{F}$ ($-M_z - 1 \le p \le N_z + 1$ and $-M_t \le q \le N_t + 1$) contains the point evaluation of the function $F(z,\ t)$ or $F(z_p,\ t_q)$.

Finally, we can determine the discrete system by substituting the expressions (3.17), (3.23), (3.27), (3.30), (3.35), (3.39), and (3.41) into (3.11) and multiplying both sides by $(h_zh_t)^{-1}$. With simplification, the discrete system to approximate the solution to (3.1)–(3.4) becomes

$$\mathcal{A}_1\mathcal{C}\mathcal{D}_{t_1} + \mathcal{D}_{s_1}\mathcal{C}\mathcal{B}_1 + \mathcal{A}_1\mathcal{C}\mathcal{D}_{t_2} + \mathcal{D}_{s_1}\mathcal{C}\mathcal{B}_2 + \mathcal{A}_3\mathcal{C}\mathcal{D}_{t_1} + \mathcal{D}_{s_3}\mathcal{C}\mathcal{B}_1 + \mathcal{A}_3\mathcal{C}\mathcal{D}_{t_2} + \mathcal{D}_{s_3}\mathcal{C}\mathcal{B}_2 + \mathcal{A}_5\mathcal{C}\mathcal{D}_{t_1} + \mathcal{D}_{s_5}\mathcal{C}\mathcal{B}_1 + \mathcal{A}_5\mathcal{C}\mathcal{D}_{t_2} + \mathcal{D}_{s_5}\mathcal{C}\mathcal{B}_2 = \mathcal{D}_{m_z}\left(\frac{1}{(\phi')^{3/2}}\right)\mathbf{F}\mathcal{D}_{m_t}\left(\frac{1}{\sqrt{\Upsilon'}}\right),$$

which leads to

$$\mathcal{A}\mathcal{C}\mathcal{D}_t + \mathcal{D}_s\mathcal{C}\mathcal{B} = \mathcal{F}, \tag{3.42}$$

where the block matrices are

$$\mathcal{A} \equiv \mathcal{A}_1 + \mathcal{A}_3 + \mathcal{A}_5 \equiv [\mathbf{a}^0_{-M_z-1}|\mathbf{A}_w|\mathbf{a}^1_{N_z+1}], \tag{3.43}$$

$$\mathcal{D}_s \equiv \mathcal{D}_{s_1} + \mathcal{D}_{s_3} + \mathcal{D}_{s_5} \equiv \left[\mathbf{b}^0_{-M_z-1}\left|I_z^{(0)}\mathcal{D}_{n_z}\left(\frac{1}{(\phi')^{5/2}}\right)\right|\mathbf{b}^1_{N_z+1}\right], \tag{3.44}$$

$$\mathcal{F} \equiv \mathcal{D}_{m_z}\left(\frac{1}{(\phi')^{3/2}}\right)\mathbf{F}\mathcal{D}_{m_t}\left(\frac{1}{\sqrt{\Upsilon'}}\right). \tag{3.45}$$

Also the $m_t \times m_t$ block matrices

$$\mathcal{D}_t \equiv \mathcal{D}_{t_1} + \mathcal{D}_{t_2} \equiv \left[\begin{array}{c|c} \mathcal{D}_{n_t}\left(\frac{1}{\sqrt{\Upsilon'}}\right) & \mathbf{O}_{n_t\times 1} \\ \hline \multicolumn{2}{c}{\mathbf{b}^T_\infty} \end{array}\right]\mathcal{D}_{m_t}(A_v), \tag{3.46}$$

$$\mathcal{B} \equiv \mathcal{B}_1 + \mathcal{B}_2 \equiv \left[\begin{array}{c} \mathbf{B}^T_w \\ \hline \mathbf{a}^T_\infty \end{array}\right]. \tag{3.47}$$

There are various methods for solving the generalized Sylvester equation (3.42). They are described in [12]. Using Theorem 2.2, (3.42) is algebraically equivalent to the system

$$\mathcal{G}\mathrm{co}(\mathcal{C}) = \mathrm{co}(\mathcal{F}), \tag{3.48}$$

where the matrix $\mathcal{G}$ involving Kronecker products is given by an $(m_z m_t) \times (m_z m_t)$ matrix

$$\mathcal{G} = \mathcal{D}_t^T \otimes \mathcal{A} + \mathcal{B}^T \otimes \mathcal{D}_s,$$

and $\mathrm{co}(\mathcal{C})$ and $\mathrm{co}(\mathcal{F})$ are $(m_z m_t) \times 1$ column vectors.

IV. PARAMETER SELECTIONS FOR THE FULLY SINC-GALERKIN METHOD

The matrices that comprise the discrete system in the Sinc-Galerkin method are full matrices. More sinc grid points lead to larger matrices and make for an expensive computation. Some cases found in [12] show how to choose an appropriate sinc grid in space and time and those selections will be used here. If the exact solution satisfies the condition

$$|w(z, t)| \le Cz^{\alpha}(1 - z)^{\beta}t^{\gamma}e^{-\delta t} \tag{4.1}$$

for $(z, t) \in (0, 1) \times (0, \infty)$, we should make the selections

$$N_z = \left[\left|\frac{\alpha}{\beta}M_z + 1\right|\right], \qquad M_t = \left[\left|\frac{\alpha}{\gamma}M_z + 1\right|\right], \qquad N_t = \left[\left|\frac{1}{h}\ln\left(\frac{\alpha}{\delta}M_z h\right) + 1\right|\right], \tag{4.2}$$

where $[|\cdot|]$ denotes the greatest integer operation, $h \equiv h_z = h_t$ and

$$h = \left(\frac{\pi d}{\alpha M_z}\right)^{1/2}. \tag{4.3}$$

For a given problem with a known real or complex solution, one can determine α, β, γ, and δ using (4.1). Then (4.2) and (4.3) provide the computational parameters. In practice, one sets $\alpha = \beta = \gamma = 1$ and $d = \pi/2$. Then from (4.2) and (4.3), $M_z = N_z = M_t$ and $h = \pi/\sqrt{2M_z}$, respectively. Numerical experiments suggest the choice $N_t = (1/2)M_z$ for the infinite time interval instead of that given in (4.2). To illustrate the performance of the method, we define $\|u_{\mathcal{S}}\|$, $\|v_{\mathcal{S}}\|$, and $\|E_{\mathcal{S}}\|$ for reporting error and convergence results between a true solution $w(z, t) = u(z, t) + iv(z, t)$ and a Sinc-Galerkin approximate solution $w_a(z, t) = u_a(z, t) + iv_a(z, t)$ on the sinc grid $\mathcal{S}$ with $h \equiv h_z = h_t$ as

$$\mathcal{S} = \left\{(z_j, t_k) : z_j = \frac{e^{jh}}{1 + e^{jh}},\ t_k = e^{kh},\ -M_z - 1 \le j \le N_z + 1,\ -M_t \le k \le N_t + 1\right\},$$

$$\|u_{\mathcal{S}}\| = \max_{\mathcal{S}}\{|u_a(z_j, t_k) - u(z_j, t_k)|\},\ \|v_{\mathcal{S}}\| = \max_{\mathcal{S}}\{|v_a(z_j, t_k) - v(z_j, t_k)|\},$$

$$\|E_{\mathcal{S}}\| = \max\{\|u_{\mathcal{S}}\|, \|v_{\mathcal{S}}\|\}.$$

We also report results on the uniform grid $\mathcal{U}$ with step size $l_z = .01$ and $l_t = 0.1$ as

$$\mathcal{U} = \{(z_m, t_n) : z_m = ml_z,\ t_n = nl_t,\ m = 0, 1, \ldots, 100,\ n = 0, 1, \ldots, 100\},$$

$$\|u_{\mathcal{U}}\| = \max_{\mathcal{U}}\{|u_a(z_m, t_n) - u(z_m, t_n)|\}, \qquad \|v_{\mathcal{U}}\| = \max_{\mathcal{U}}\{|v_a(z_m, t_n) - v(z_m, t_n)|\},$$

$$\|E_{\mathcal{U}}\| = \max\{\|u_{\mathcal{U}}\|, \|v_{\mathcal{U}}\|\}.$$

For visual clarity, all three-dimensional graphs of approximate solutions are plotted on a uniform grid $\mathbf{U}$ with step sizes of 0.025 in z and 0.1 in t as

$$\mathbf{U} = \{(z_m, t_n) : z_m = (0.025)m,\ t_n = (0.1)n,\ m = 0, 1, \ldots, 40,\ n = 0, 1, \ldots, 100\}. \tag{4.4}$$

Two-dimensional graphs of approximate solutions are plotted on a uniform grid $\mathbf{U}_z$ with step size of 0.025 in z as

$$\mathbf{U}_z = \{z_m = (0.025)m,\ m = 0, 1, \ldots, 40\}. \tag{4.5}$$

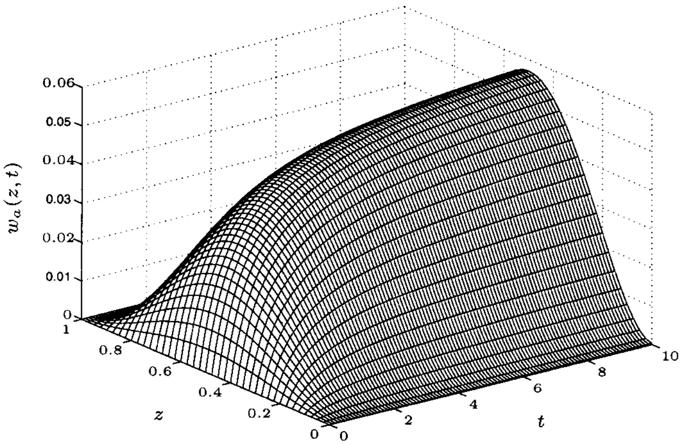

FIG. 5. The graph of the approximate solution $w_a(z, t)$ on the uniform grid **U** for Example 5.1 with $A_v(t) = (t + 1)/(t + 2)$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

V. NUMERICAL EXAMPLE FOR TIME-INDEPENDENT BOUNDARY CONDITIONS

Example 5.1. In this example we choose $A_v(t) = (t + 1)/(t + 2)$ and the parameter $\sigma = 1$. This is a heat equation with a time-dependent coefficient, Neumann boundary condition at $z = 0$ and mixed boundary condition at $z = 1$. So the initial-boundary-value problem becomes

$$\frac{\partial w(z, t)}{\partial t} - \left(\frac{t+1}{t+2}\right)\frac{\partial^2 w(z, t)}{\partial z^2} = F(z, t), \qquad 0 < z < 1, \qquad 0 < t$$

$$\frac{\partial w(0, t)}{\partial z} = 0, \qquad 0 < t$$

$$w(1, t) + \frac{\partial w(1, t)}{\partial z} = 0, \qquad 0 < t$$

$$w(z, 0) = 0, \qquad 0 < z < 1.$$

With the forcing function given by

$$F(z, t) = \frac{(z^2 - 2z^3 + z^4)}{(1 + t)^2} - (1 - 6z + 6z^2)\,\frac{2t}{(1 + t)},$$

the exact solution is

$$w(z, t) = z^2(1 - z)^2\,\frac{t}{(1 + t)}.$$

The discrete system given by (3.48) is solved to produce the approximate solution in Fig. 5. A time plot of the approximate solution is graphed for each of $z = 0$, .20, .50, and 1 in Fig. 6.

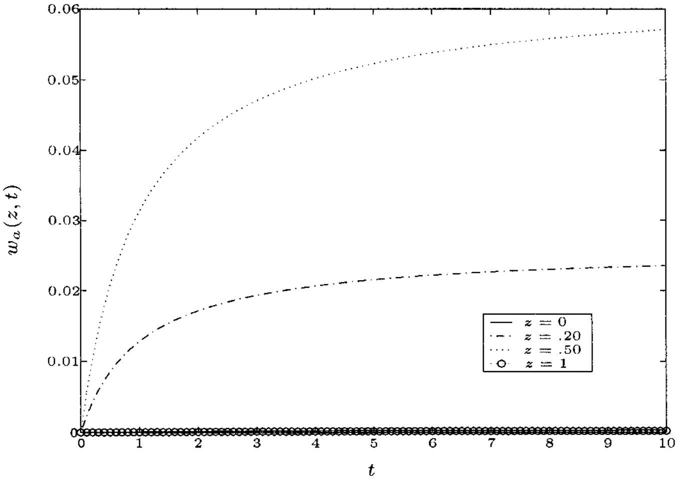

FIG. 6. The graph of the approximate solution $w_a(z, t)$ at $z = 0, 0.20, 0.50, 1$ on the uniform grid $\mathbf{U}_z$ for Example 5.1 with $A_v(t) = (t + 1)/(t + 2)$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

Note that the solutions at $z = 0$ and $z = 1$ are both zero for this example. The numerical results given in Table I demonstrate the convergence of the Sinc-Galerkin method as the parameter M_z is repeatedly doubled. Note that the discrete system (3.48) involves no imaginary part, so both $v(z, t)$ and $v_a(z, t)$ are identically zero. Hence $\|u_{\mathcal{S}}\| = \|E_{\mathcal{S}}\|$ and (though not shown) $\|u_{\mathcal{U}}\| = \|E_{\mathcal{U}}\|$.

VI. SOLVING THE PROBLEM WITH TIME-DEPENDENT BOUNDARY CONDITIONS

More generally, a Sinc-Galerkin method can be developed to produce a discrete system for a partial differential equation with time-dependent boundary conditions similar to (3.1)–(3.4). In this section we consider the special case of (1.1)–(1.4) with $k = 1$:

$$\mathcal{L}w(z, t) - iw(z, t) = F(z, t), \qquad 0 < z < 1, \qquad 0 < t, \tag{6.1}$$

subject to the time-dependent boundary conditions and the initial condition

$$\frac{\partial w(0, t)}{\partial z} = 0, \qquad 0 < t, \tag{6.2}$$

TABLE I. Errors on the sinc grid $\mathcal{S}$ and the uniform grid $\mathcal{U}$ for the choices $M_z = N_z = M_t = 2N_t$ for Example 5.1 with $A_v(t) = (t + 1)/(t + 2)$ and $\sigma = 1$.

M_z	N_z	M_t	N_t	h	$\|u_{\mathcal{S}}\|$	$\|v_{\mathcal{S}}\|$	$\|E_{\mathcal{S}}\|$	$\|E_{\mathcal{U}}\|$
4	4	4	2	1.111	2.021e−02	0.000e+00	2.021e−02	1.799e−02
8	8	8	4	0.785	4.124e−03	0.000e+00	4.124e−03	3.560e−03
16	16	16	8	0.555	3.949e−04	0.000e+00	3.949e−04	3.334e−04
32	32	32	16	0.393	1.227e−05	0.000e+00	1.227e−05	1.026e−05

$$w(1, t) + \sigma A_v(t) \frac{\partial w(1, t)}{\partial z} = 0, \qquad 0 < t, \tag{6.3}$$

$$w(z, 0) = 0, \qquad 0 < z < 1. \tag{6.4}$$

We rewrite the additional boundary basis function $w_1(z)$ in (3.6) as

$$w_1(z, t) = w_g(z) + \sigma A_v(t) w_h(z), \tag{6.5}$$

where

$$w_g(z) = (1 - z)z^2, \qquad w_h(z) = (3 - 2z)z^2.$$

The approximate solution $w_a(z, t)$ can be written in the separated form

$$w_a(z, t) = w_c(z, t) + w_b(z, t), \tag{6.6}$$

where

$$w_c(z, t) = \sum_{j=-M_z-1}^{N_z+1} \sum_{k=-M_t}^{N_t+1} c_{jk} \xi_j^*(z) \zeta_k(t),$$

$$w_b(z, t) = \sigma A_v(t) \sum_{k=-M_t}^{N_t+1} c_{(N_z+1)k} w_h(z) \zeta_k(t), \tag{6.7}$$

and the basis functions $\xi_j^*(z)$ are

$$\xi_j^*(z) = \begin{cases} w_0(z), & \text{if } j = -M_z - 1 \\ \left(\dfrac{S_j}{\phi'}\right)(z), & \text{if } j = -M_z, \ldots, N_z \\ w_g(z), & \text{if } j = N_z + 1, \end{cases}$$

with $\zeta_k(t)$ in (3.9).

The unknown coefficients c_{jk} are determined by orthogonalizing the residual with respect to the set of sinc basis functions in (3.5). This yields the discrete system

$$(\mathscr{L}w_c, S_p S_q^*) + (\mathscr{L}w_b, S_p S_q^*) - i(w_a, S_p S_q^*) = (F, S_p S_q^*). \tag{6.8}$$

The first inner product $(\mathscr{L}w_c, S_p S_q^*)$ in (6.8) is similar to the inner product in (3.11). Letting p, q range over all values gives the matrix approximation to $(\mathscr{L}w_c, S_p S_q^*)$,

$$h_z h_t \{\mathscr{A}_g \mathscr{C} \mathscr{D}_t + \mathscr{D}_g \mathscr{C} \mathscr{B}\}, \tag{6.9}$$

where the $m_z \times m_z$ block matrices are

$$\mathcal{A}_g \equiv [\mathbf{a}^0_{-M_z-1}|\mathbf{A}_w|\mathbf{a}^g_{N_z+1}],$$

$$\mathcal{D}_g \equiv \left[\mathbf{b}^0_{-M_z-1}\left|I_z^{(0)}\mathcal{D}_{n_z}\left(\frac{1}{(\phi')^{5/2}}\right)\right|\mathbf{b}^g_{N_z+1}\right]. \tag{6.10}$$

The block matrix $\mathcal{D}_t$ is in (3.46) and $\mathcal{B}$ is in (3.47) and the $m_z \times 1$ column vectors $\mathbf{a}^0_{-M_z-1}$ and $\mathbf{b}^0_{-M_z-1}$ are given in (3.21). The $m_z \times 1$ column vectors $\mathbf{a}^g_{N_z+1}$ and $\mathbf{b}^g_{N_z+1}$ have pth component, $p = -M_z - 1, \ldots, N_z + 1$,

$$[\mathbf{a}^g_{N_z+1}]_p \equiv -\frac{w_g''(z_p)}{(\phi'(z_p))^{3/2}}, \qquad [\mathbf{b}^g_{N_z+1}]_p \equiv \frac{w_g(z_p)}{(\phi'(z_p))^{3/2}}. \tag{6.11}$$

Next, the second inner product in (6.8) is

$$(\mathcal{L}w_b, S_pS_q^*) = \sigma \sum_{k=-M_t}^{N_t} c_{(N_z+1)k}\{(w_h(A_v\zeta_k)', S_pS_q^*) - (A_v^2\zeta_k w_h'', S_pS_q^*)\}$$
$$+ \sigma c_{(N_z+1)(N_t+1)}\{(w_h(A_v w_\infty)', S_pS_q^*) - (A_v^2 w_\infty w_h'', S_pS_q^*)\}. \tag{6.12}$$

Applying integration by parts once with respect to t to the first inner product in (6.12), the result is

$$(w_h(A_v\zeta_k)', S_pS_q^*) = \int_0^\infty \int_0^1 w_h(z)(A_v(t)\zeta_k(t))' S_p(z)S_q^*(t)\sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dzdt$$
$$= \int_0^1 \frac{w_h(z)S_p(z)}{\sqrt{\phi'(z)}}\left[B_{T_3} - \int_0^\infty A_v(t)\zeta_k(t)(S_q^*\sqrt{\Upsilon'})'(t)dt\right]dz.$$

The boundary condition $B_{T_3} = A_v(t)\zeta_k(t)S_q^*(t)\sqrt{\Upsilon'(t)}\big|_0^\infty$ equals zero. Applying the sinc quadrature rule in space and time (2.5), then leads to

$$(w_h(A_v\zeta_k)', S_pS_q^*) = \int_0^1 \frac{w_h(z)S_p(z)}{\sqrt{\phi'(z)}}\int_0^\infty -A_v(t)\zeta_k(t)(S_q^*\sqrt{\Upsilon'})'(t)dtdz$$
$$\approx h_z h_t \frac{w_h(z_p)}{(\phi'(z_p))^{3/2}}\left\{-\frac{1}{h_t}\delta_{qk}^{(1)} + \frac{1}{2}\delta_{qk}^{(0)}\right\}A_v(t_k)\sqrt{\Upsilon'(t_k)}. \tag{6.13}$$

The rest of the inner products in (6.12) are directly integrated by the sinc quadrature rule in space and time (2.5) as

$$(w_h''A_v^2\zeta_k, S_pS_q^*) = \int_0^\infty \int_0^1 w_h''(z)A_v^2(t)\zeta_k(t)S_p(z)S_q^*(t)\sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dzdt \approx h_z h_t \frac{w_h''(z_p)}{(\phi'(z_p))^{3/2}}\frac{A_v^2(t_k)}{\sqrt{\Upsilon'(t_k)}},$$

$$(w_h(A_v w_\infty)',\ S_p S_q^*) = \int_0^\infty \int_0^1 w_h(z)(A_v(t)w_\infty(t))' S_p(z) S_q^*(t) \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dzdt$$
$$\approx h_z h_t \frac{w_h(z_p)}{(\phi'(z_p))^{3/2}} \left\{ \frac{w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}} A_v'(t_q) + \frac{w_\infty'(t_q)}{\sqrt{\Upsilon'(t_q)}} A_v(t_q) \right\},$$

and

$$(w_h'' A_v^2 w_\infty,\ S_p S_q^*) = \int_0^\infty \int_0^1 w_h''(z) A_v^2(t) w_\infty(t) S_p(z) S_q^*(t) \sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\, dzdt$$
$$\approx h_z h_t \frac{w_h''(z_p)}{(\phi'(z_p))^{3/2}} \frac{A_v^2(t_q) w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}}. \quad (6.14)$$

Therefore, (6.12) becomes

$$(\mathcal{L}w_b,\ S_p S_q^*) \approx h_z h_t \sigma \sum_{k=-M_t}^{N_t} \left[\frac{w_h(z_p)}{(\phi'(z_p))^{3/2}} c_{(N_z+1)k} \left\{ -\frac{1}{h_t} \delta_{qk}^{(1)} + \frac{1}{2} \delta_{qk}^{(0)} \right\} A_v(t_k) \sqrt{\Upsilon'(t_k)} \right.$$
$$\left. - \frac{w_h''(z_p)}{(\phi'(z_p))^{3/2}} c_{(N_z+1)k} \frac{A_v^2(t_k)}{\sqrt{\Upsilon'(t_k)}} \right] + h_z h_t \sigma \left[\frac{w_h(z_p)}{(\phi'(z_p))^{3/2}} c_{(N_z+1)(N_t+1)} \left\{ \frac{w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}} A_v'(t_q) + \frac{w_\infty'(t_q)}{\sqrt{\Upsilon'(t_q)}} A_v(t_q) \right\} \right.$$
$$\left. - \frac{w_h''(z_p)}{(\phi'(z_p))^{3/2}} c_{(N_z+1)(N_t+1)} \frac{A_v^2(t_q) w_\infty(t_q)}{\sqrt{\Upsilon'(t_q)}} \right]. \quad (6.15)$$

Letting p, q range over all values gives the matrix approximation to $(\mathcal{L}w_b,\ S_p S_q^*)$,

$$h_z h_t \sigma \{ \mathcal{A}_b \mathcal{C} \mathcal{D}_{t_b} + \mathcal{D}_{s_b} \mathcal{C} \mathcal{B}_b \}, \quad (6.16)$$

where the $m_z \times m_z$ block matrices

$$\mathcal{A}_b \equiv [\mathbf{O}_{m_z \times 1} | \mathbf{O}_{m_z \times n_z} | \mathbf{a}_{N_z+1}^h],$$

$$\mathcal{D}_{s_b} \equiv [\mathbf{O}_{m_z \times 1} | \mathbf{O}_{m_z \times n_z} | \mathbf{b}_{N_z+1}^h], \quad (6.17)$$

the $m_t \times m_t$ block matrices

$$\mathcal{D}_{t_b} = \mathcal{D}_t \mathcal{D}_{m_t}(A_v), \qquad \mathcal{B}_b \equiv \left[\frac{\mathbf{B}_w^T}{\mathbf{a}_\infty^T} \right] \mathcal{D}_{m_t}(A_v) + \left[\frac{\mathbf{O}_{n_t \times m_t}}{\mathbf{b}_\infty^T} \right] \mathcal{D}_{m_t}(A_v'), \quad (6.18)$$

$\mathcal{D}_t$ is in (3.46), and $\mathbf{a}_\infty$ and $\mathbf{b}_\infty$ are in (3.29). The $m_z \times 1$ column vectors $\mathbf{a}_{N_z+1}^h$, $\mathbf{b}_{N_z+1}^h$ have pth component, $p = -M_z - 1, \ldots, N_z + 1$,

$$[\mathbf{a}^h_{N_z+1}]_p \equiv -\frac{w_h''(z_p)}{(\phi'(z_p))^{3/2}}, \qquad [\mathbf{b}^h_{N_z+1}]_p \equiv \frac{w_h(z_p)}{(\phi'(z_p))^{3/2}}. \tag{6.19}$$

In the Sinc-Galerkin method, the solution $w_a(z, t)$ in (6.7) is usually evaluated at the nodal points (z_p, t_q), where $z_p = e^{ph}/(1 + e^{ph})$ and $t_q = e^{qh}$. So letting p, q range over all values gives the $m_z \times m_t$ evaluator matrix approximation to $w_a(z_p, t_q)$,

$$\{\mathcal{A}_{e_a}\mathcal{C}\mathcal{D}_{e_a} + \sigma\mathcal{A}_{e_b}\mathcal{C}\mathcal{D}_{e_b}\}, \tag{6.20}$$

where the $m_z \times m_z$ block matrices

$$\mathcal{A}_{e_a} \equiv \left[\mathbf{e}^0_{-M_z-1}\middle| I_z^{(0)}\mathcal{D}_{n_z}\left(\frac{1}{\phi'}\right)\middle|\mathbf{e}^g_{N_z+1}\right],$$

$$\mathcal{A}_{e_b} \equiv [\mathbf{O}_{m_z\times 1}|\mathbf{O}_{m_z\times n_z}|\mathbf{e}^h_{N_z+1}]. \tag{6.21}$$

The $m_t \times m_t$ block matrices

$$\mathcal{D}_{e_a} \equiv \left[\frac{I_{n_t\times n_t}|\mathbf{O}_{n_t\times 1}}{\mathbf{e}^T_\infty}\right], \qquad \mathcal{D}_{e_b} \equiv \mathcal{D}_{e_a}\mathcal{D}_{m_t}(A_v), \tag{6.22}$$

and the $m_z \times 1$ column vectors, $\mathbf{e}^0_{-M_z-1}$, $e^g_{N_z+1}$, $\mathbf{e}^h_{N_z+1}$ have pth component, $-M_z-1 \le p \le N_z+1$,

$$[\mathbf{e}^0_{-M_z-1}]_p \equiv w_0(z_p), \qquad [\mathbf{e}^g_{N_z+1}]_p \equiv w_g(z_p), \qquad [\mathbf{e}^h_{N_z+1}]_p \equiv w_h(z_p). \tag{6.23}$$

The $m_t \times 1$ column vector, $\mathbf{e}_\infty$ has qth component, $-M_t \le q \le N_t + 1$,

$$[\mathbf{e}_\infty]_q \equiv w_\infty(t_q). \tag{6.24}$$

Next, applying the sinc quadrature rule (2.5), the inner product $(w_a, S_pS_q^*)$ in (6.8) is

$$(w_a, S_pS_q^*) = \int_0^\infty \int_0^1 w_a(z, t)S_p(z)S_q^*(t)\sqrt{\frac{\Upsilon'(t)}{\phi'(z)}}\,dzdt \approx h_zh_t\,\frac{1}{(\phi'(z_p))^{3/2}}\,w_a(z_p, t_q)\,\frac{1}{\sqrt{\Upsilon'(t_q)}}, \tag{6.25}$$

where $w_a(z_p, t_q)$ is in (6.20). Letting p, q range over all values gives the matrix approximation to $(w_a, S_pS_q^*)$,

$$h_zh_t\{\mathcal{D}_g\mathcal{C}\mathcal{D}_{t^*} + \sigma\mathcal{D}_{s_b}\mathcal{C}\mathcal{D}_t\}, \tag{6.26}$$

where $\mathcal{D}_t$ is in (3.46), $\mathcal{D}_g$ is in (6.10), $\mathcal{D}_{s_b}$ is in (6.17), and $\mathcal{D}_{t^*} = \mathcal{D}_t\mathcal{D}(1/A_v)$. Using the expressions (6.9), (6.16), (6.26), and (3.41) substitute into (6.8) to arrive at the discrete system

$$\mathcal{A}_g\mathcal{C}\mathcal{D}_t + \mathcal{D}_g\mathcal{C}\mathcal{B} + \sigma\mathcal{A}_b\mathcal{C}\mathcal{D}_{t_b} + \sigma\mathcal{D}_{s_b}\mathcal{C}\mathcal{B}_b - i\mathcal{D}_g\mathcal{C}\mathcal{D}_{t^*} - i\sigma\mathcal{D}_{s_b}\mathcal{C}\mathcal{D}_t = \mathcal{F}, \tag{6.27}$$

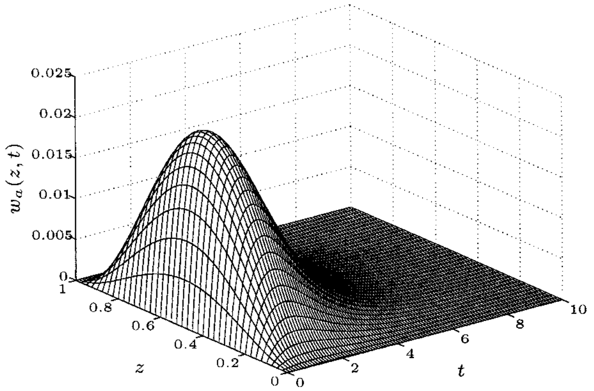

FIG. 7. The graph of the approximate solution $w_a(z, t)$ on the uniform grid **U** for Example 7.1 with $A_v(t) = 2 - e^{-2t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

where $\mathcal{A}_g$, $\mathcal{D}_g$ are in (6.10), $\mathcal{F}$ is in (3.45), $\mathcal{B}$ is in (3.47), $\mathcal{D}_t$ is in (3.46), $\mathcal{A}_b$, $\mathcal{D}_{s_b}$ are in (6.17), and $\mathcal{D}_{t_b}$, $\mathcal{B}_b$ are in (6.18). With Theorem 2.2, (6.27) is algebraically equivalent to the linear discrete system

$$\mathcal{G}\mathrm{co}(\mathcal{C}) = \mathrm{co}(\mathcal{F}), \tag{6.28}$$

where the matrix $\mathcal{G}$ involving Kronecker products is given by an $(m_z m_t) \times (m_z m_t)$ matrix

$$\mathcal{G} = \mathcal{D}_t^T \otimes \mathcal{A}_g + \mathcal{B}^T \otimes \mathcal{D}_g + \mathcal{D}_{t_b}^T \otimes \sigma\mathcal{A}_b + \mathcal{B}_b^T \otimes \sigma\mathcal{D}_{s_b} - \mathcal{D}_{t^*}^T \otimes i\mathcal{D}_g - \mathcal{D}_t^T \otimes i\sigma\mathcal{D}_{s_b},$$

and $\mathrm{co}(\mathcal{C})$ and $\mathrm{co}(\mathcal{F})$ are $(m_z m_t) \times 1$ vectors.

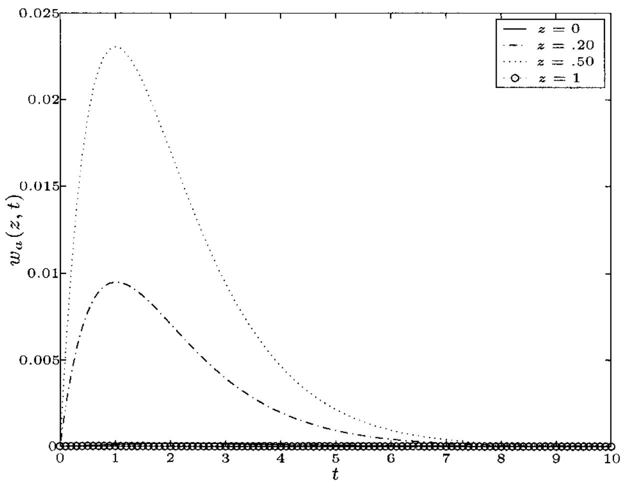

FIG. 8. The graph of the approximate solution $w_a(z, t)$ at $z = 0, 0.20, 0.50, 1$ on the uniform grid $\mathbf{U}_z$ for Example 7.1 with $A_v(t) = 2 - e^{-2t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

TABLE II. Errors on the sinc grid $\mathcal{S}$ and the uniform grid $\mathcal{U}$ for the choices $M_z = N_z = M_t = 2N_t$ for Example 7.1 with $A_v(t) = 2 - e^{-2t}$ and $\sigma = 1$.

M_z	N_z	M_t	N_t	h	$\|u_{\mathcal{S}}\|$	$\|v_{\mathcal{S}}\|$	$\|E_{\mathcal{S}}\|$	$\|E_{\mathcal{U}}\|$
4	4	4	2	1.111	5.525e−03	4.165e−03	5.525e−03	5.925e−03
8	8	8	4	0.785	1.067e−03	9.711e−04	1.067e−03	1.286e−03
16	16	16	8	0.555	1.273e−04	8.918e−05	1.273e−04	1.278e−04
32	32	32	16	0.393	3.711e−06	2.912e−06	3.711e−06	3.842e−06

VII. NUMERICAL EXAMPLES FOR TIME-DEPENDENT BOUNDARY CONDITIONS

Example 7.1. An increasing time-dependent function $A_v(t) = 2 - e^{-2t}$ and $\sigma = 1$ are chosen in (6.1)–(6.4). Then the partial differential equation with time-dependent boundary conditions becomes

$$\frac{\partial w(z,t)}{\partial t} - (2 - e^{-2t})\frac{\partial^2 w(z,t)}{\partial z^2} - iw(z,t) = F(z,t), \qquad 0 < z < 1, \qquad 0 < t$$

$$\frac{\partial w(0,t)}{\partial z} = 0, \qquad 0 < t$$

$$w(1,t) + (2 - e^{-2t})\frac{\partial w(1,t)}{\partial z} = 0, \qquad 0 < t$$

$$w(z,0) = 0, \qquad 0 < z < 1.$$

With the forcing function given by

$$F(z,t) = (z - z^2)^2(1 - t)e^{-t} - (2 - e^{-2t})(2(1 - 2z)^2te^{-t} - 4(z - z^2)te^{-t}) - i(z - z^2)^2te^{-t},$$

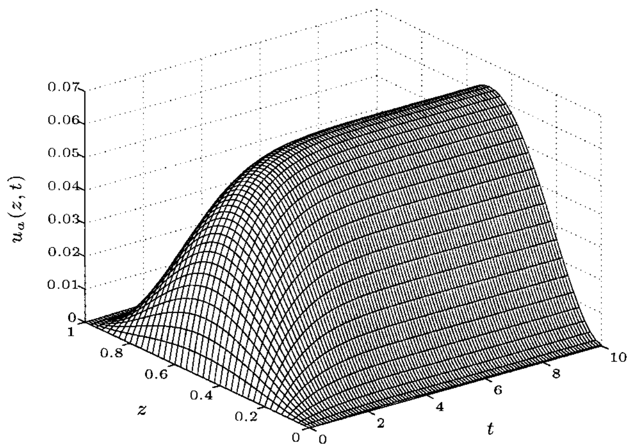

FIG. 9. The graph of the real part of the approximate solution, $u_a(z, t)$, on the uniform grid **U** for Example 7.2 with $A_v(t) = 1 + te^{1-t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

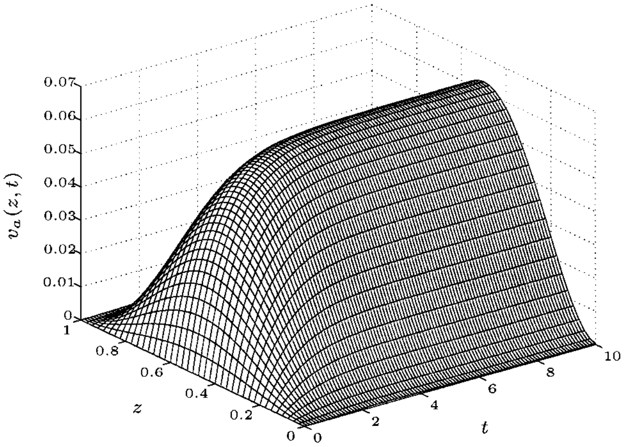

FIG. 10. The graph of the imaginary part of the approximate solution, $v_a(z, t)$, on the uniform grid **U** for Example 7.2 with $A_v(t) = 1 + te^{1-t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

the real-valued solution is

$$w(z, t) = (z - z^2)^2 te^{-t}.$$

Note that the solution is real, so the imaginary part, $v(z, t)$, is zero. The discrete system given by (6.28) is solved for the approximate solution shown in Fig. 7. A time plot of the approximate solution is graphed for each of $z = 0$, 0.20, 0.50, and 1 in Fig. 8. Again the graphs at $z = 0$ and $z = 1$ are both zero. Both the numerical errors on the sinc grid $\mathcal{S}$ and the uniform grid $\mathcal{U}$ are reported in Table II. Note that the discrete system (6.28) includes an imaginary part. So, though $v(z, t) \equiv 0$, $v_a(z, t)$ is only zero to the accuracy of the method. Thus there is error involved in

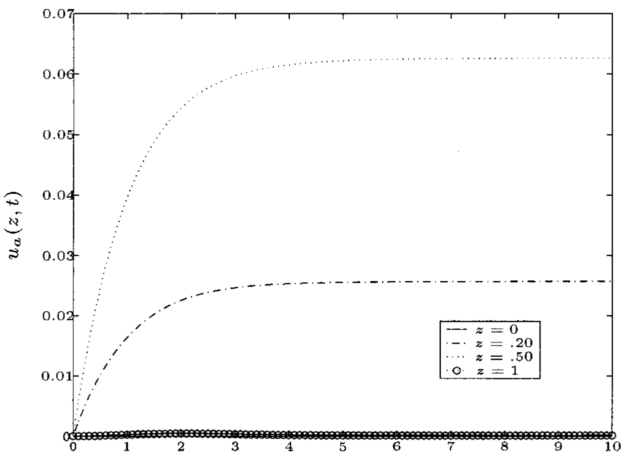

FIG. 11. The graph of the real part of the approximate solution, $u_a(z, t)$, at $z = 0$, 0.20, 0.50, 1 on the uniform grid $\mathbf{U}_z$ for Example 7.2 with $A_v(t) = 1 + te^{1-t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

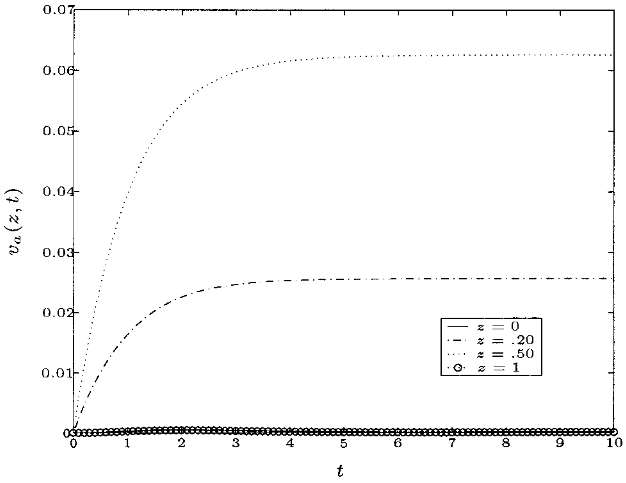

FIG. 12. The graph of the imaginary part of the approximate solution, $v_a(z, t)$, at $z = 0, 0.20, 0.50, 1$ on the uniform grid $\mathbf{U}_z$ for Example 7.2 with $A_v(t) = 1 + te^{1-t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

the approximate $v_a(z, t)$ which is reflected in the column $\|v_{\mathcal{S}}\|$. In all cases this is smaller than $\|u_{\mathcal{S}}\|$.

Example 7.2. With the time-dependent function $A_v(t) = 1 + te^{1-t}$ and $\sigma = 1$ in (6.1)–(6.4) the partial differential equation with time-dependent boundary condition becomes

$$\frac{\partial w(z,t)}{\partial t} - (1 + te^{1-t})\frac{\partial^2 w(z,t)}{\partial z^2} - iw(z,t) = F(z,t), \qquad 0 < z < 1, \qquad 0 < t$$

$$\frac{\partial w(0,t)}{\partial z} = 0, \qquad 0 < t$$

$$w(1,t) + (1 + te^{1-t})\frac{\partial w(1,t)}{\partial z} = 0, \qquad 0 < t$$

$$w(z,0) = 0, \qquad 0 < z < 1.$$

With the forcing function given by

TABLE III. Errors on the sinc grid $\mathcal{S}$ and the uniform grid $\mathcal{U}$ for $M_z = N_z = M_t = 2N_t$ for Example 7.2 with $A_v(t) = 1 + te^{1-t}$ and $\sigma = 1$.

M_z	N_z	M_t	N_t	h	$\|u_{\mathcal{S}}\|$	$\|v_{\mathcal{S}}\|$	$\|E_{\mathcal{S}}\|$	$\|E_{\mathcal{U}}\|$
4	4	4	2	1.111	6.767e−03	2.566e−02	2.566e−02	2.574e−02
8	8	8	4	0.785	1.258e−03	5.085e−03	5.085e−03	5.335e−03
16	16	16	8	0.555	1.268e−04	5.100e−04	5.100e−04	5.126e−04
32	32	32	16	0.393	4.102e−06	1.559e−05	1.559e−05	1.583e−05

TABLE IV. Errors on the sinc grid $\mathcal{S}$ and the uniform grid $\mathcal{U}$ for the choices $M_z = N_z = M_t = 2N_t$ for Example 7.3 with $A_v(t) = 1 + 3te^{1-t}$ and $\sigma = 1$.

M_z	N_z	M_t	N_t	h	$\|u_{\mathcal{S}}\|$	$\|v_{\mathcal{S}}\|$	$\|E_{\mathcal{S}}\|$	$\|E_{\mathcal{U}}\|$
4	4	4	2	1.111	1.948e−02	2.278e−02	2.278e−02	2.285e−02
8	8	8	4	0.785	5.138e−03	3.953e−03	5.138e−03	5.197e−03
16	16	16	8	0.555	4.755e−04	4.542e−04	4.755e−04	4.919e−04
32	32	32	16	0.393	1.504e−05	1.396e−05	1.504e−05	1.511e−05

$$F(z, t) = (z^2(1 - z)^2 + iz^2(1 - z)^2)e^{-t} - (1 + te^{1-t})((2 - 12z + 12z^2) + i(2 - 12z + 12z^2))(1 - e^{-t}) - i(z^2(1 - z)^2 + iz^2(1 - z)^2)(1 - e^{1-t}),$$

the complex-valued solution is

$$w(z, t) = (z^2(1 - z)^2 + iz^2(1 - z)^2)(1 - e^{-t}).$$

The purpose of this example is to illustrate the approximate solution of a complex-valued partial differential equation with a nonzero steady state. The discrete system given by (6.28) is solved for the approximate complex-valued solution. Both real and imaginary parts of the approximate solution are shown in Figs. 9 and 10, respectively. Of course, the real and imaginary parts of this solution are the same, so the approximations are very similar. Time plots of the real and imaginary parts of the approximate solution are graphed for each of $z = 0$, 0.2, 0.50, and 1 in Figs. 11 and 12, respectively. Again, the approximations are nearly identical and again the graphs at $z = 0$ and $z = 1$ are zero. The numerical errors are reported in Table III and bear out the fact that the real and imaginary approximations are accurate to the same order of magnitude. The results here are remarkably similar to those reported in Table II for Example 7.1 in spite of the fact that this example has a complex-valued solution. The results are also similar to those in Table I for Example 5.1, which involved a real-valued solution to the problem with a

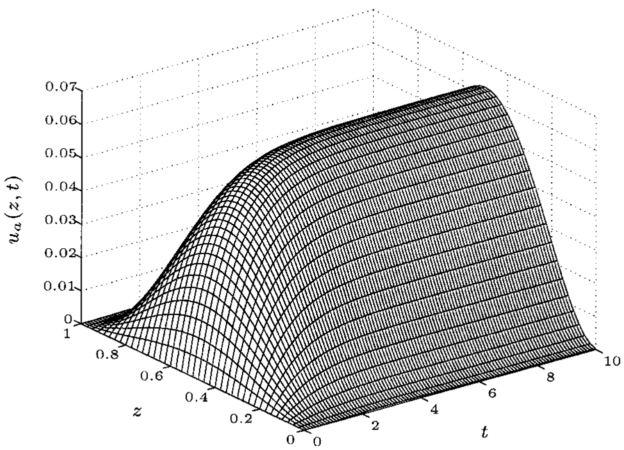

FIG. 13. The graph of the approximate solution (real part) $u_a(z, t)$ on the uniform grid **U** for Example 7.3 with $A_v(t) = 1 + 3te^{1-t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

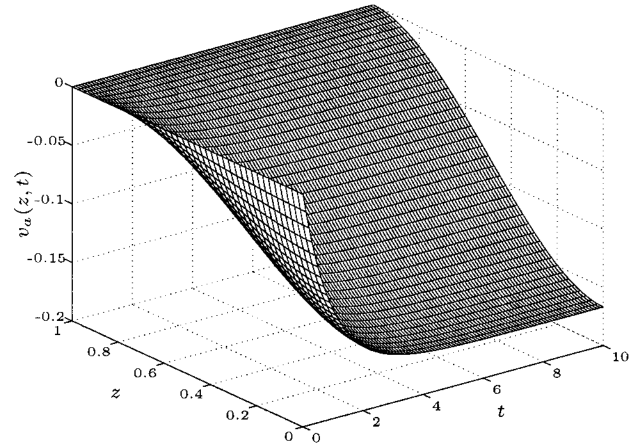

FIG. 14. The graph of the approximate solution (imaginary part) $v_a(z, t)$ on the uniform grid **U** for Example 7.3 with $A_v(t) = 1 + 3te^{1-t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

time-independent boundary condition. These results indicate that the method is performing extremely well under various circumstances.

Example 7.3. Consider the partial differential equation with time-dependent boundary condition in (6.1)–(6.4) with $A_v(t) = 1 + 3te^{1-t}$ and $\sigma = 1$ given by

$$\frac{\partial w(z, t)}{\partial t} - (1 + 3te^{1-t}) \frac{\partial^2 w(z, t)}{\partial z^2} - iw(z, t) = F(z, t), \qquad 0 < z < 1, \qquad 0 < t$$

$$\frac{\partial w(0, t)}{\partial z} = 0, \qquad 0 < t$$

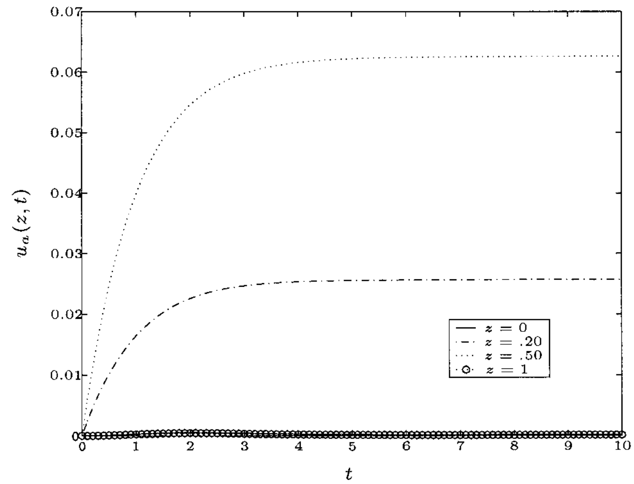

FIG. 15. The graph of the approximate solution (real part) $u_a(z, t)$ at $z = 0, 0.20, 0.50, 1$ on the uniform grid $\mathbf{U}_z$ for Example 7.3 with $A_v(t) = 1 + 3te^{1-t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16$, $N_t = 8$.

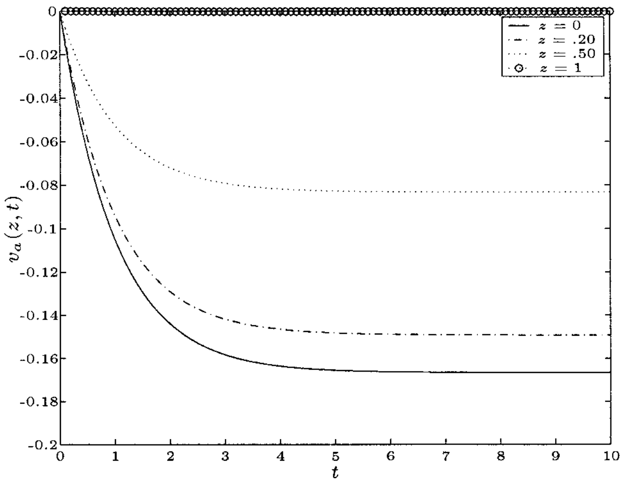

FIG. 16. The graph of the approximate solution (imaginary part) $v_a(z, t)$ at $z = 0, 0.20, 0.50, 1$ on the uniform grid $\mathbf{U}_z$ for Example 7.3 with $A_v(t) = 1 + 3te^{1-t}$ and $\sigma = 1$ with $M_z = N_z = M_t = 16, N_t = 8$.

$$w(1, t) + (1 + 3te^{1-t}) \frac{\partial w(1, t)}{\partial z} = 0, \qquad 0 < t$$

$$w(z, 0) = 0, \qquad 0 < z < 1.$$

With the forcing function

$$F(z, t) = \left(z^2(1 - z)^2 + \frac{i}{6}(3z^2 - 2z^3 - 1)\right)e^{-t} - (1 + 3te^{1-t})((2 - 12z + 12z^2) + i(1 - 2z))(1 - e^{-t}) - i\left(z^2(1 - z)^2 + \frac{i}{6}(3z^2 - 2z^3 - 1)\right)(1 - e^{-t}),$$

the complex-valued solution is

$$w(z, t) = \left(z^2(1 - z)^2 + \frac{i}{6}(3z^2 - 2z^3 - 1)\right)(1 - e^{-t}).$$

This example illustrates a solution whose real and imaginary parts behave in a significantly different manner. The real part is an increasing function of time and the imaginary part is a decreasing function of time. In addition, the steady state is nonzero. All these are possible complications for the method. The discrete system given by (6.28) is solved for the numerical solution. The errors are reported in Table IV. Again these results are remarkably similar to the previous examples. This demonstrates that the method is quite robust, seemingly impervious to different behaviors of the true solution. The graphical results for the real and imaginary parts of the approximate solution are shown in Figs. 13 and 14, respectively. Time plots for the real and imaginary parts of the approximate solution are graphed for each of $z = 0, 0.2, 0.50$, and 1 in Figs. 15 and 16, respectively. The time plot of the real part in Fig. 15 shows that the approximate solutions at $z = 0$ and $z = 1$ are both zero. The time plot of the imaginary part in Fig. 16 shows that the approximate solution at $z = 1$ is zero.

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