Selected Solutions for Section 3.5

Problem 3, page 220 – 221.
Find the extreama of $f(x,y) = 3x+2y$ subject to the constraint that $g(x,y) = 2x^2+3y^2 = 3$.
Solution
Note that $\nabla f(x,y) = (3,2)$ and $\nabla g(x,y) = (4x,6y)$. Using the Lagrange Multiplier Method, we want to find (x,y) and λ such that the following equations are satisfied:

$$\begin{aligned} 3 &= \lambda(4x) && (1) \\ 2 &= \lambda(6y) && (2) \\ 2x^2+3y^2 &= 3. && (3) \end{aligned}$$

Solving (1) for x in terms of λ and solving (2) for y in terms of λ, we have $x = \frac{3}{4\lambda}$ and $x = \frac{1}{3\lambda}$. Plugging these relations into (3), we have an equation in terms of λ. That is,

$$\frac{9}{8\lambda^2} + \frac{1}{3\lambda^2} = 3.$$

Solving this equation for λ, we find that $\lambda = \pm\sqrt{\frac{35}{72}} = \pm\frac{1}{6}\sqrt{\frac{35}{2}}$. This leads us to

$$x = \pm\frac{9}{\sqrt{70}}$$

and

$$y = \pm\frac{4}{\sqrt{70}}.$$

Hence, the points where extreme values are possible are $(\pm\frac{9}{\sqrt{70}}, \pm\frac{4}{\sqrt{70}})$. By examining the linear structure of f, we see that the point $(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}})$ will yield the smallest function value. Therefore $f(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}}) = -\frac{35}{\sqrt{70}} = -\frac{\sqrt{70}}{2}$ is the minimum function value. Similarly, $f(\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}}) = \frac{35}{\sqrt{70}} = \frac{\sqrt{70}}{2}$ is the maximum function value.

NOTE: Problem 7 is done in a very similar manner, only there are 4 equations to solve since f, g are functions of x, y, z for that problem. Hence, the gradients have 3 components and provide 3 equations for the Lagrange multiplier λ.

Problem 5.
Find the extrema of $f(x, y) = xy$ subject to the constraint that $g(x, y) = x + y = 1$.
Solution
Note that $\nabla f(x, y) = (y, x)$ and $\nabla g(x, y) = (1, 1)$. Find (x, y) and λ which satisfy the following equations.

$$\begin{aligned} y &= \lambda && (4) \\ x &= \lambda && (5) \\ x + y &= 1. && (6) \end{aligned}$$

Equations (4) and (5) tell us that $y = \lambda = x$. Substituting this relation into (6) tells us that $2x = 1$ or $x = \frac{1}{2}$. Therefore, $y = \frac{1}{2}$, and the point $(\frac{1}{2}, \frac{1}{2})$ gives us the only extremum of $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.

Problem 15.
Find the extrema of $f(x, y) = xy$ subject to the constraints that $g(x, y) = 2x + 3y \leq 10$, $x \geq 0$, $y \geq 0$.
Solution
Note that we are finding the minimum of the function f over the closed region shown in Figure 1. Furthermore, $f \geq 0$ for all (x, y) in this region.

Step 1 Find the critical points of f. That is, find (x, y) so that $\nabla f(x, y) = 0$. Hence, we have the equations $y = 0$ and $x = 0$. Then $(0, 0)$ is the only critical point of f, and $f(0, 0) = 0$.

Step 2 Using Lagrange Multiplier Method, find the critical points of f on the boundary of the region shown in Figure 1. We seek to find (x, y) and λ satisfying

$$\begin{aligned} y &= 2\lambda && (7) \\ x &= 2\lambda && (8) \\ 2x + 3y &= 10. && (9) \end{aligned}$$

Substituting (7) and (8) into (9) and solving for λ, we have $\lambda = \frac{5}{6}$. Substituting this into (7) and (8) gives us the point $(\frac{5}{2}, \frac{5}{3})$. Hence, we have two extrema: $f(0, 0) = 0$ is the minimum value, and $f(\frac{5}{2}, \frac{5}{3}) = \frac{25}{6}$ is the maximum value.

Problem 19

A parcel delivery service requires that the dimensions of a rectangular box be such that the length plus twice the width plus twice the height be no more than 108 inches. What is the volume of the largest-volume box the company will deliver?

Solution

Half of the problem is just getting this question stated in a manner that we know how to answer. First, let's draw a picture of our box. We use the variables l, w, h for the lenth, width and height measured in inches, see Figure 2. The volume of the box is $V(l, w, h) = lwh$, and the constraint given in the problem is that $g(l, w, h) = l + 2w + 2h \leq 108$. Hence, the problem can be stated as follows:

Find the maximum value of $V = lwh$ subject to the constraint that $g(l, w, h) \leq 108$. Hence, we are finding the extrema of the function V over the region defined by $g(l, w, h) \leq 108$.

Step 1 Find the critical points of V. That is, find (l, w, h) so that $\nabla V = (wh, lh, lw) = \mathbf{0}$. The critical points are all of the form $(l, 0, 0)$, $(0, w, 0)$ and $(0, 0, h)$. Note that none of these critical points will give us a maximum value of V since the volume will always be 0 at any of these points.

Step 2 Lagrange Multiplier Method. Find (l, w, h) and λ so that

$$wh = \lambda \qquad (10)$$

$$lh = 2\lambda \qquad (11)$$

$$lw = 2\lambda \qquad (12)$$

$$l + 2w + 2h = 108 \qquad (13)$$

Equations (11) and (12) tell us that $lh = 2\lambda = lw$, there $l(h - w) = 0$. Since l can't be 0, then we must have that $h = w$. Now, we need to express the variable l in terms of either h or w. Equation (10) tells us that $\lambda = h^2$, and using this relationship in (11) gives us that $lh = 2h^2$. Therefore $h(l - 2h) = 0$, and we must have $l = 2h$. Then substituting for l and w in terms of h into (13) gives us that

$$2h + 2h + 2h = 108 \quad \Rightarrow h = 18.$$

Then we know that $w = 18$ and $l = 36$. Hence the extreme value of the volume V occurs at the point $(36, 18, 18)$, and the maximum volume is $V(36, 18, 18) = 11,664\text{in}^2$. Hence, the largest-volume box that the company will deliver is one with the dimensions $l = 36$ inches, $w = h = 18$ inches, and the volume is given above.

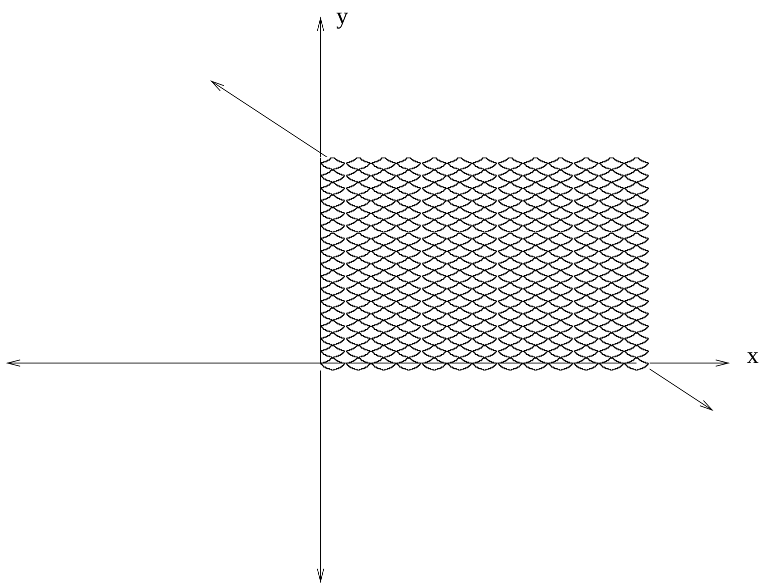

Figure 1: In Problem 15, The closed region over which we seek extrema of f.

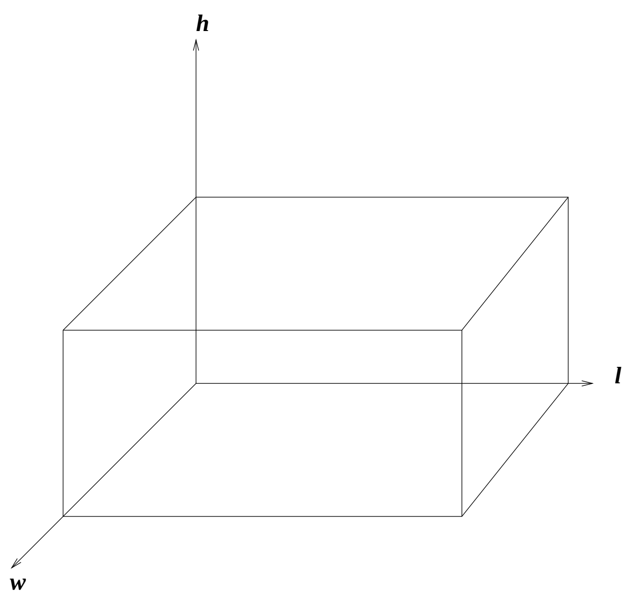

Figure 2: In Problem 19, Sample Box.