MATH 224 TEST 3

Carefully Read and Follow Directions, and show all work in the space provided below. If you require extra space, clearly label your work on a clean sheet of paper and attach it to the back of your test. No credit will be given for unsubstantiated answers.

1. Set up (but **DO NOT EVALUATE**) an iterated integral for $\int\int_W x^3y^2dA$, where W is the region bounded by $y = x$, $y = x^2$, $x = 0$ and $x = 1$. (*12 pts*)
   Either of the following integrals are acceptable:

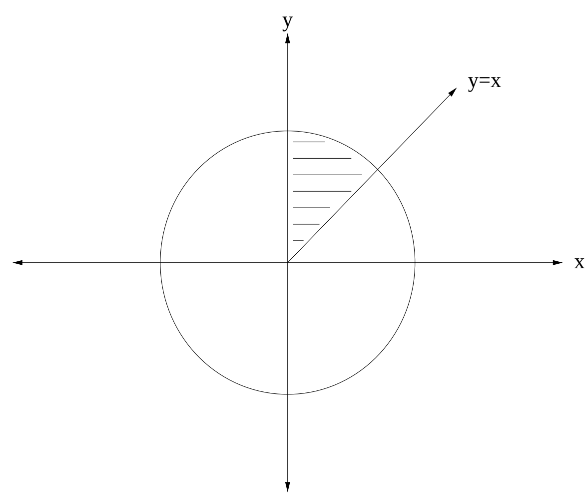
$$\int\int_W x^3y^2dA = \int_0^1 \int_{x^2}^{x} x^3y^2dydx = \int_0^1 \int_y^{\sqrt{y}} x^3y^2dxdy$$

2. Use polar coordinates to set up (but **DO NOT EVALUATE**) the integral $\int\int_W (x^2 + y^2)dA$ where W is the region in the **first quadrant** lying inside $x^2 + y^2 = 1$ and above $y = x$. Sketch the region W. (*15 pts*)

   Bounds are given by: $0 \leq r \leq 1$ and $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$. And the integrand has the following form in polar coordinates: $x^2 + y^2 = r^2$. Hence, the integral is given by either of the following:

$$\int\int_W (x^2 + y^2)dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r^2rdrd\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r^3drd\theta \quad \text{or}$$

$$\int\int_W (x^2 + y^2)dA = \int_0^1 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^3d\theta dr$$



3. Given the integral $\int_0^1 \int_y^1 e^x dxdy$, sketch the region of integration. Then interchange the order of integration, and evaluate the integral. (*20 pts*)

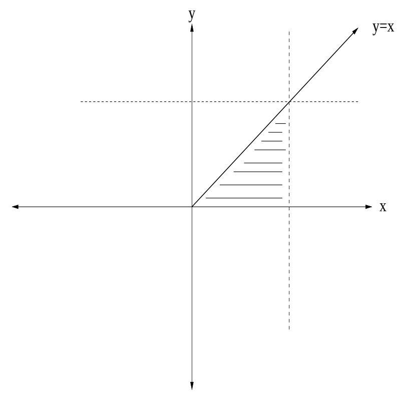
The bounds of the original integral are:

$$y \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq 1.$$

And the new bounds are given by:

$$0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq x.$$

A sketch of the region of integration is given below:



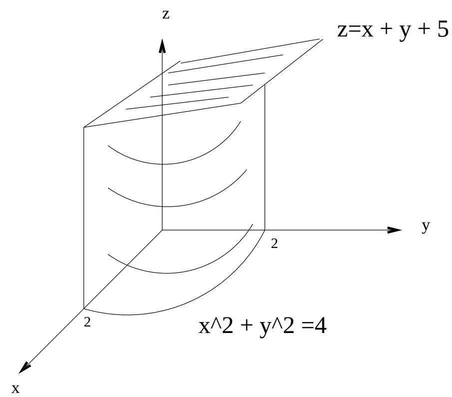
Hence, we have

$$\begin{aligned}
\int_0^1 \int_y^1 e^x dxdy &= \int_0^1 \int_0^x e^x dydx \\
&= \int_0^1 xe^x dx \\
&= xe^x \Big|_0^1 - \int_0^1 e^x dx \quad \textit{Int. by Parts} \\
&= e - (e-1) \\
&= 1
\end{aligned}$$

4. Evaluate the integral $\int\int\int_W (2x+3y+z)dxdydz$ where the region of integration is given by $W = [0,2]\times[-1,1]\times[0,1]$. (*15 pts*)

$$\begin{aligned}
\int\int\int_W (2x+3y+z)dxdydz &= \int_0^1\int_{-1}^1\int_0^2 (2x+3y+z)dxdydz \\
&= \int_0^1\int_{-1}^1 (3y+z)x + x^2 \Big|_{x=0}^{x=2} dydz \\
&= \int_0^1\int_{-1}^1 6y+2z+4dydz \\
&= \int_0^1 3y^2 + (2z+4)y \Big|_{y=-1}^{y=1} dz \\
&= \int_0^1 (8+4z)dz \\
&= 8z + 2z^2 \Big|_{z=0}^{z=1} \\
&= 10
\end{aligned}$$

5. (a) Use **cartesian** coordinates to set up (but **DO NOT EVALUATE**) the integral which describes the volume lying inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$ and below the plane $z - x - y = 5$. (*8 pts*)



Above is a sketch of the region, and the bounds of the region are given by:

$$-2 \le x \le 2, \quad -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \quad 0 \le z \le x + y + 5.$$

Therefore, the integral is given by

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{x+y+5} dz dy dx.$$

(b) Use **cylindrical** coordinates to set up (but **DO NOT EVALUATE**) the integral which describes the volume as given above in part (a). (*8 pts*)

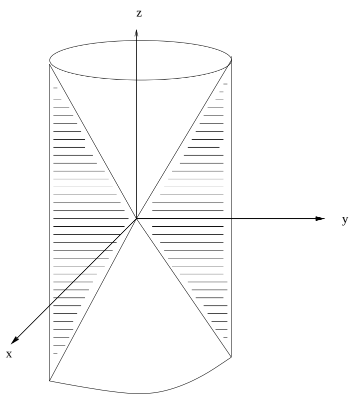
In cylindrical coordinates, the bounds of integration are given by

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le r(\cos\theta + \sin\theta) + 5.$$

The volume integral is then given by

$$V = \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{r(\cos\theta+\sin\theta)+5} .$$

6. Use **cylindrical** coordinates to set up (but **DO NOT EVALUATE**) the integral which describes the volume of the solid bounded outside the cone $z^2 = x^2 + y^2$ and inside the cylinder $x^2 + y^2 = 1$ lying above the plane $z = -1$ and below the plane $z = 1$. (*12 pts*)



The figure above is a sketch of the solid described here. The bounds for cylindrical coordinates are given by:

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad -r \le z \le r.$$

The volume integral is given by

$$V = \int_0^{2\pi} \int_0^1 \int_{-r}^{r} r dz dr d\theta.$$

Many students attempted to find the volume by beginning with the bounds $-1 \le z \le 1$. If one approaches the problem in this manner, then the volume needs to be written as the sum of two integrals: one integral for the region $-1 \le z \le 0$ and another for the region $0 \le z \le 1$. The reason for this is that the corresponding bounds for the radius variable, r, must be given in terms of z. In particular, $z = -r$ (or $r = -z$) for $-1 \le z \le 0$, and $z = r$ for $0 \le z \le 1$. Recall that the variable r represents a distance; therefore, one cannot use the bound $z \le r \le 1$ in conjunction with the bounds $-1 \le z \le 1$ since this would allow for negative values of r. If two integrals are used, then one can use the bounds $-1 \le z \le 0$ with $-z \le r \le 1$ and $0 \le z \le 1$ with $z \le r \le 1$. One way to get around this would be to use the bounds $|z| \le r \le 1$ with $-1 \le z \le 1$ and use only one integral. Of course, during the process of integration, these bounds would result in a splitting of the integrals as well.

7. Find the volume of the right hemisphere described by $x^2 + y^2 + z^2 = 4$ and $y \geq 0$. *(10 pts)*
In spherical coordinates, the bounds of integration are given by

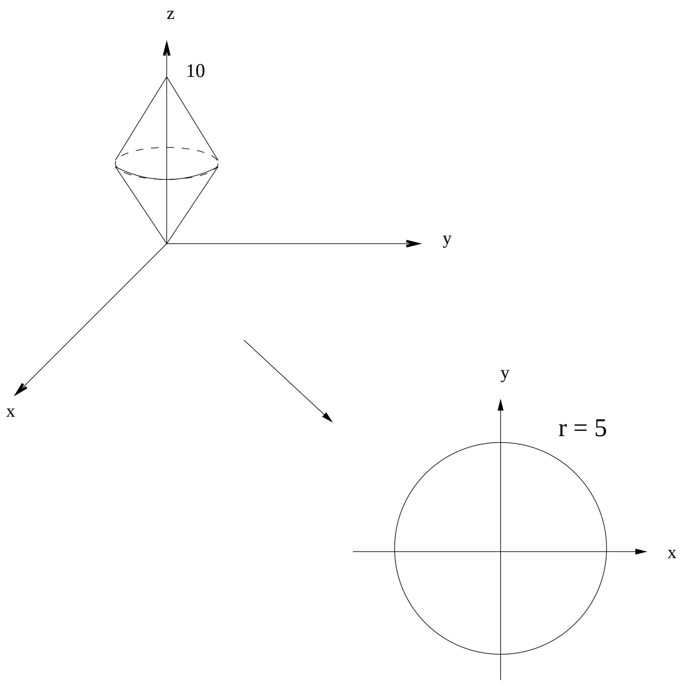
$$0 \leq \rho \leq 1, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq \pi.$$

And the volume integral is given by

$$\begin{aligned} V &= \int_0^2 \int_0^\pi \int_0^\pi \rho^2 \sin\phi \, d\phi d\theta d\rho \\ &= \int_0^2 \rho^2 \int_0^\pi -\cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta d\rho \\ &= \int_0^2 2\rho^2 \int_0^\pi d\theta d\rho \\ &= \int_0^2 2\pi\rho^2 d\rho \\ &= \frac{2\pi}{3}\rho^3 \Big|_{\rho=0}^{\rho=2} = \frac{16\pi}{3} \end{aligned}$$

BONUS QUESTION: (*5 pts*)
Find the volume of the region enclosed by the cones $z = \sqrt{x^2 + y^2}$ and $z = 10 - \sqrt{x^2 + y^2}$.

A sketch of the region along with a sketch of the level curve at the intersection, $z = 5$, is given by



The bounds for integration in cylindrical coordinates are given by

$$0 \le r \le 5, \quad 0 \le \theta \le 2\pi, \quad r \le z \le 10 - r.$$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^5 \int_r^{10-r} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^5 10r - 2r^2\,dr\,d\theta \\ &= \int_0^{2\pi} 5r^2 - \frac{2}{3}r^3 \Big|_0^5 \, d\theta \\ &= 2\pi\left(\frac{125}{3}\right) = \frac{250\pi}{3} \end{aligned}$$