Romberg Integration

Recall Composite Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{(b-a)}{12} h^2 f''(c), \quad c \in (a,b)$$

where $f \in C^2[a,b]$, $h = \frac{(b-a)}{n}$, $x_j = a + jh$, for $j = 0,1,\ldots n-1$

Romberg Integration:

① Use Comp. Trap. Rule to obtain rough approximation of $\int_a^b f(x)dx$.

② Then apply Richardson's Extrapolation to obtain more accurate approximations from the original computations. (avoids using a fancy quadrature rule).

First, notation for repeated application of the Comp. Trap. Rule:

Ⓐ Write Comp. Trap Rule as:

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{n-1} f(x_j)\right] - \frac{(b-a)}{12} h^2 f''(c)$$

k	# of SubIntervals	Nodes	Step-Sizes
0	$m_0 = 1$	x_0, x_1	$h_0 = \frac{b-a}{1}$
1	$m_1 = 2$	x_0, x_1, x_2	$h_1 = \frac{b-a}{2}$
2	$m_2 = 4$	x_0, x_1, x_2, x_3, x_4	$h_2 = \frac{b-a}{4}$
$\vdots$	$\vdots$		$\vdots$
k	$m_k = 2^k$	$x_0, x_1, \ldots, x_{m_k}$	$h_k = \frac{b-a}{m_k}$

For any pair m_k and h_k, the Comp. Trap. Rule looks like:

$$\int_a^b f(x)dx = \frac{h_k}{2}\left[f(a)+f(b)+2\sum_{j=1}^{m_k-1} f(x_j)\right] - \frac{(b-a)}{12} h_k^2 f''(c_k), \text{ with } c_k \in (a,b).$$

Since $h_{k+1} = \frac{1}{2}h_k$, then we are simply inserting new nodes without throwing out old ones, so we do not want to repeat function evaluations in step $k+1$ that we obtained during step k.

(c) Derive the relation which allows us to use the info from the previous step. (see pgs. 3,4 : Transparencies)

Define:

$$R(0,0) = \frac{h_0}{2}[f(a)+f(b)] = \frac{b-a}{2}[f(a)+f(b)]$$

Eqn (10)
$$R(n,0) = \frac{1}{2}\Big[\underbrace{R(n-1,0)}_{\substack{\text{info from}\\(n-1)\text{st step}}} + h_n \big[\underbrace{\sum_{i=1}^{2^{n-1}} f(a+(2i-1)h_n)}_{\text{new nodes}}\big]\Big], \text{ for } n=1,2,3,\ldots$$

See pg 213-215

- This allows us to reuse our old info and only perform the new function evaluations required.

- Eqn (10) allows us to compute the 1st column of our Romberg Integration Table!

Recursive Trapezoid Rule and Romberg Integration

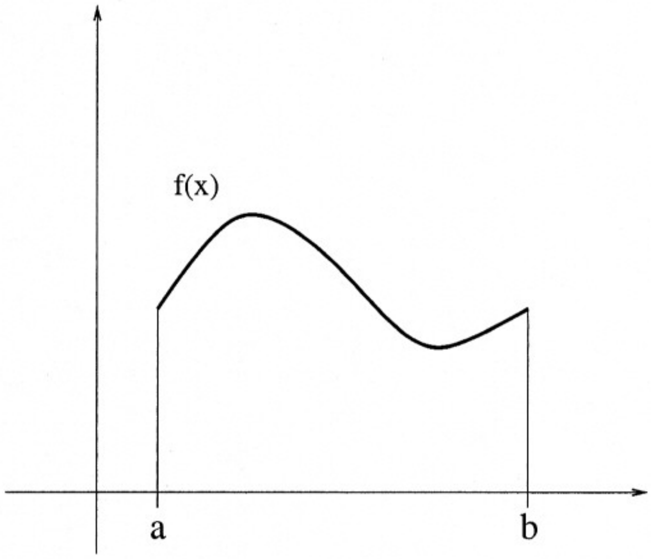

Figure 1: Trapezoidal Rule with $m_0 = 1$

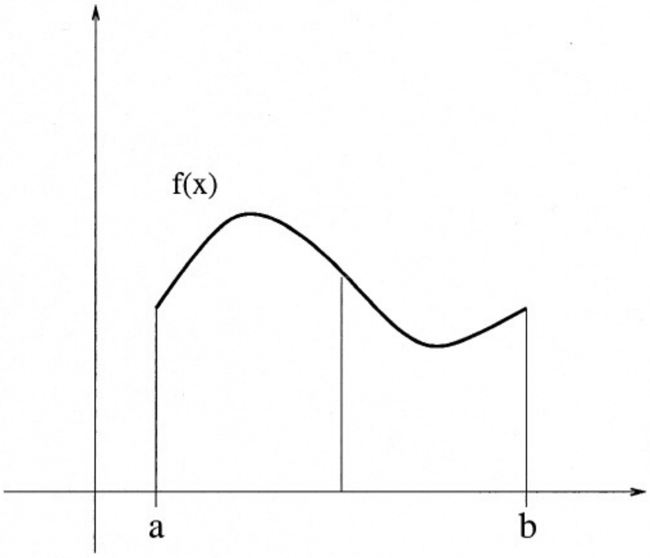

Figure 2: Trapezoidal Rule with $m_1 = 2$

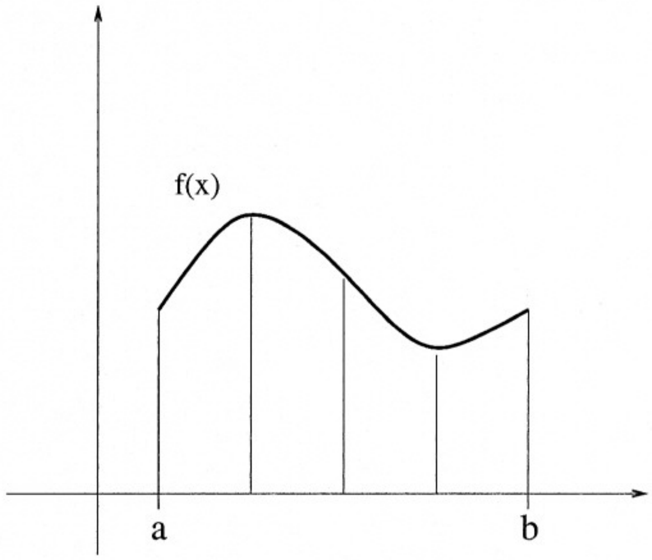

Figure 3: Trapezoidal Rule with $m_2 = 4$

① Apply Richardson's Extrapolation to our 1st column

Notation $R(k,j) = R_{k,j}$

$$R_{k,j} = \frac{4^j}{4^j-1} R_{k,j-1} - \frac{1}{4^j-1} R_{k-1,j-1}$$

for $k=1,2,\ldots,n$ and $j=1,2,\ldots,k$ with $R_{k,j}$ being an $O(h_k^{2j})$ approximation to the integral.

→ Recall that the Numerical Implementation uses.

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^j-1}\left[R_{k,j-1} - R_{k-1,j-1}\right]$$

Remarks: ① Stopping Criteria:

If the # of ROWS of the TABLE (n) is not predetermined, then we stop when

$|R_{n-1,n-1} - R_{n,n}| < \epsilon$ and $|R_{n-2,n-2} - R_{n-1,n-1}| < \epsilon$,

Where ϵ is some prescribed tolerance.

② Same warnings apply as in Richardson's Extrapolation for Numerical Diff.

③ Uses low-order technique of integration & extrapolation ideas to obtain higher-order approximations without performing more function evals.

Example 1: Using Romberg Integration, approximate

$$\int_0^2 2^x dx = \frac{3}{\ln 2} \approx 4.3281$$

k	h_k	$R(k,0)$	$R(k,1)$	$R(k,2)$	$R(k,3)$	$R(k,4)$	$R(k,5)$	...		...	$R(k,9)$
0	2.0	2.5000									
1	1.0	3.2500	3.5000								
2	0.5	3.7463	3.9118	3.9392							
3	0.25	4.0264	4.1198	4.1336	4.1367						
4	0.125	4.1745	4.2239	4.2309	4.2324	4.2328					
5	0.0625	4.2506	4.2760	4.2795	4.2802	4.2804	4.2805				
6	0.0312	4.2892	4.3020	4.3038	4.3042	4.3043	4.3043	4.3043			
7	0.0156	4.3086	4.3151	4.3159	4.3161	4.3162	4.3162	4.3162	4.3162		
8	0.0078	4.3183	4.3216	4.3220	4.3221	4.3221	4.3221	4.3221	4.3221	4.3221	
9	0.0039	4.3232	4.3248	4.3250	4.3251	4.3251	4.3251	4.3251	4.3251	4.3251	4.3251

Example 2: Using Romberg Integration, approximate

$$\int_0^{\pi} \sin x dx = 2$$

k	h_k	$R(k,0)$	$R(k,1)$	$R(k,2)$	$R(k,3)$	$R(k,4)$
1	π	0.0000				
2	$\pi/2$	1.5708	2.0944			
3	$\pi/4$	1.8961	2.0046	1.9986		
4	$\pi/8$	1.9742	2.0003	2.0000	2.0000	
5	$\pi/16$	1.9936	2.0000	2.0000	2.0000	2.0000

7.

```
% driver2.m
% This driver creates a matrix and performs Romberg integration
% on the function in f.m

% Define Dimension
n = 4;

% Define Interval [a,b]
% Must edit! This is the interval over which we
% are attempting to approximate the integral of
% the function in f.m
a = 1;
b = 1.5;

% Initialize h_k vector which stores the
% interval width for the kth iteration
% of the trapezoid rule.
h = zeros(n,1);
h(1) = b-a;

% Define Matrix M which contains
% the values of the Romberg integration.
M      = zeros(n,n);
M(1,1) = 0.5*(f(a) + f(b));

for k = 2:n
 % k is the row that we are computing.
 h(k) = h(k-1)/2;
 % temp stores the set of values at which we
 % must evalute the function f
 temp = a + h(k)*[1:2:2^(k-1)-1];
 M(k,1) =  (M(k-1,1) + h(k-1)*sum(f(temp)))/2;

 for j = 2:k
    % j is the number of the column within row k
    M(k,j) =  M(k,j-1) + (M(k,j-1)-M(k-1,j-1))/(4^(j-1)-1);
 end; %end=j-loop

end; %end-k-loop
```