

**MATH 442 MidTerm Exam**

**Carefully Read and Follow Directions** Clearly label your work and attach it to this sheet. No credit will be given for unsubstantiated answers.

1. (Paper and Pencil, no Matlab!) Let  $S(x)$  be defined by

$$S(x) = \begin{cases} x^3 + ax^2 + 3x, & 0 \leq x \leq 1 \\ bx^3 + (6 + 2a - 3b)x - 1, & 1 \leq x \leq 2 \end{cases}$$

- (a) Determine values of  $a$  and  $b$  which make  $S(x)$  a cubic spline.  
 (b) Using the values in part (a), is  $S(x)$  a natural cubic spline?
2. You may use the Matlab function *csfit.m* and the driver *csdriver.m* for this problem. **NOTE: You should not attempt to modify the function file.** Only modify the **driver** for this exercise. Currently, the driver *csdriver.m* computes the clamped cubic spline that passes through the points  $\{(x_k, f(x_k))\}_{k=0}^3$ , on the graph of

$$f(x) = x + \frac{2}{x},$$

using the nodes  $x_0 = 0.5, x_1 = 1.0, x_2 = 1.5, x_3 = 2.0$ . It uses the first derivative boundary conditions  $S'(x_0) = f'(x_0)$  and  $S'(x_3) = f'(x_3)$ . Finally, it graphs  $f(x)$  and the clamped cubic spline interpolant  $S(x)$  on the same coordinate system.

Your assignment is to modify *csdriver.m* to compute the clamped cubic spline that passes through the points  $\{(x_k, f(x_k))\}_{k=0}^3$ , on the graph of  $f(x) = \cos(x^2)$ , using the nodes  $x_0 = 0, x_1 = \sqrt{\pi/2}, x_2 = \sqrt{3\pi/2}, x_3 = \sqrt{5\pi/2}$ . Use the first derivative boundary conditions  $S'(x_0) = f'(x_0)$  and  $S'(x_3) = f'(x_3)$ . Graph  $f(x)$  and the clamped cubic spline interpolant  $S(x)$  on the same coordinate system, and label the graph accordingly.

3. Let  $f(x) = e^x$ . You may use **numdiff.m** to perform the calculations for this problem. You can also download **ex2.m** to see how this script works for  $f(x) = \sin x$  to approximate  $f'(0.8)$ . When you are ready to record your work, create a table which summarizes the results of parts a,b and c. The table should have the following form.

$h$	$\approx f'(2.3)$	True Error	Error Bound
0.1			
0.01			
0.001			

- (a) Calculate approximations for  $f'(2.3)$  using the Centered Difference Formula

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

with  $h = 0.1$ ,  $h = 0.01$  and  $h = 0.001$ . Carry eight or nine decimal places.

- (b) Compare with  $f'(2.3) = e^{2.3}$ .

- (c) Compute bounds for the truncation error. For each case, use

$$|f'''(\xi)| \leq e^{2.4} \approx 11.02317638$$

4. (a) Use Taylor Series expansions to derive the following order  $O(h^4)$  formula for approximating  $f'(x)$ .

$$f'(x) = \frac{1}{12h} [-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)] + O(h^4) \quad (1)$$

- (b) If you performed your calculations correctly in Problem 3, you saw the error terms decrease by a factor of  $10^{-2}$  with each iteration (decrease in step size). Since a centered difference formula with truncation error of order  $O(h^2)$  is applied, then **each time the step size is decreased by a factor of one tenth, 0.1, one should observe a decrease in error by a factor of  $(0.1)^2 = 0.01 = 10^{-2}$** . You might want to check your work to verify that this is true. Now, the formula you derived in part (a) has truncation error of order  $O(h^4)$ .

- Write a brief statement describing the amount of decrease in error that one should expect to observe when applying the difference approximation in equation (1) when the step size is decreased by a factor of one tenth at each successive iteration.
- Modify the mfile **numdiff.m** in order to implement the difference formula indicated in equation (1). (HINT: First copy everything from numdiff.m into a new file called **midnum4.m** and save it. Then modify midnum4.m so that it performs the appropriate calculations.) Using the same function as in Problem Number 3 and a beginning step size of 1.0, verify that your approximations are correct and verify the truncation error of order 4. Submit the results of the output in the table format. (you can put this in a text file, edit and print if you know how to use the *diary* command.

5. (a) Assuming roundoff error in the numerical calculations and truncation error in the difference formula, a bound on the *true* error,  $E(f, h)$ , for the difference calculation

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

is given by

$$|E(f, h)| \leq \frac{3\epsilon}{2h} + \frac{Mh^4}{30},$$

where

$$M = \max_{a \leq x \leq b} |f^{(5)}(x)|.$$

Verify that the value of  $h$  that minimizes the error bound is

$$h = \left(\frac{45\epsilon}{4M}\right)^{1/5}$$

- (b) Using the same example as in Problem 3 with a beginning step size of  $h = 1$ , an approximate optimal step size can be approximated. In order to obtain the value for  $\epsilon$ , type "eps" from the command window in Matlab on whatever machine you are using. Estimate the optimal step size using this information and the result of part a. Verify this with numerical results from your mfile **midnum4.m**.
6. Verify that the **degree of precision** for Simpson's Rule is 3. Consider an arbitrary finite interval  $[a, b]$ .