Appendix A

Claim: Given $2\langle Ax_1-\mu x_1, h\rangle + \langle Ah,h\rangle - \mu\langle h,h\rangle \le 0 \ \forall\, h\in\mathbb{R}^n$, it must be true that

$$\langle Ax_1-\mu x_1, h\rangle = 0 \quad \forall\, h\in\mathbb{R}^n.$$

Proof: Spse not. Spse that $\exists\, h_0\in\mathbb{R}^n$ so that

$$\langle Ax_1-\mu x_1, h_0\rangle \neq 0.$$

Case 1: If $\langle Ax_1-\mu x_1, h_0\rangle > 0$, then $2\langle Ax_1-\mu x_1, h_0\rangle > 0$.
And we know that

$$0 < 2\langle Ax_1-\mu x_1, h_0\rangle \le \underbrace{\mu\langle h_0,h_0\rangle - \langle Ah_0,h_0\rangle}_{\text{(so this is positive also)}}$$

Then we can choose $\alpha > 0$ but small enough so that

$$0 < 2\langle Ax_1-\mu x_1, h_0\rangle > \alpha\left[\mu\langle h_0,h_0\rangle - \langle Ah_0,h_0\rangle\right]$$

$\Longrightarrow$

mult. by α. $$2\alpha\langle Ax_1-\mu x_1, h_0\rangle > \alpha^2\left[\mu\langle h_0,h_0\rangle - \langle Ah_0,h_0\rangle\right]$$

$$\Longrightarrow \quad 2\langle Ax_1-\mu x_1, \alpha h_0\rangle > \mu\langle \alpha h_0,\alpha h_0\rangle - \langle A(\alpha h_0),(\alpha h_0)\rangle$$

$\Longrightarrow$

$$\underbrace{2\langle Ax_1-\mu x_1, \alpha h_0\rangle + \langle A(\alpha h_0),(\alpha h_0)\rangle - \mu\langle \alpha h_0,\alpha h_0\rangle}_{(*)} > 0$$

Hence $\alpha h_0\in\mathbb{R}^n$ so that $(*) > 0$. This contradicts our given inequality.

Case 2: If $\langle Ax_1-\mu x_1, h_0\rangle < 0$, then we break this up into 2 subcases $\longrightarrow$ see next page.

<u>Subcase 1</u>: Assume that $\langle Ax_1 - \mu x_1, h_0 \rangle < 0$ and that
$$2\langle Ax_1 - \mu x_1, h_0 \rangle = \mu \langle h_0, h_0 \rangle - \langle Ah_0, h_0 \rangle.$$

Then choose any $\alpha \in (0,1)$ and

$$\begin{aligned} 2\langle Ax_1 - \mu x_1, \alpha h_0 \rangle &= \alpha\left[2\langle Ax_1 - \mu x_1, h_0 \rangle\right] \\ &= \alpha\left[\mu\langle h_0, h_0\rangle - \langle Ah_0, h_0\right] \\ &> \alpha^2\left[\mu\langle h_0, h_0\rangle - \langle Ah_0, h_0\rangle\right] \\ &= \mu\langle \alpha h_0, \alpha h_0\rangle - \langle A(\alpha h_0), (\alpha h_0)\rangle \end{aligned}$$

Hence, we have produce $\hat{h} = \alpha h_0 \in \mathbb{R}^n$ so that

$$2\langle Ax_1 - \mu x_1 \hat{h} \rangle + \langle A\hat{h}, \hat{h} \rangle - \mu \langle \hat{h}, \hat{h} \rangle > 0$$

(Hence, we have a contradiction of our given inequality)

<u>Subcase 2</u>: Assume that $\langle Ax_1 - \mu x_1, h_0 \rangle < 0$ and that
$$2\langle Ax_1 - \mu x_1, h_0 \rangle < \mu\langle h_0, h_0 \rangle - \langle Ah_0, h_0 \rangle$$

Observe that for all h,

(1) $2\langle Ax_1 - \mu x_1, h \rangle + \langle Ah, h \rangle - \mu\langle h, h \rangle \leq 0$ and

(2) $2\langle Ax_1 - \mu x_1, -h \rangle + \langle A(-h), (-h) \rangle - \mu\langle -h, -h \rangle \leq 0$

Simplifying (2) yields (3)

(3) $-2\langle Ax_1 - \mu x_1, h \rangle + \langle Ah, h \rangle - \mu\langle h, h \rangle \leq 0 \quad \forall h \in \mathbb{R}^n$

Adding (1) to (3), we have

$$2[\langle Ah,h\rangle - \mu\langle h,h\rangle] \leq 0 \quad \forall h \in \mathbb{R}^n$$

$\Rightarrow$

$$\langle Ah,h\rangle - \mu\langle h,h\rangle \leq 0 \quad \forall h \in \mathbb{R}^n$$

$\Rightarrow$

$$\mu\langle h,h\rangle - \langle Ah,h\rangle \geq 0 \quad \forall h \in \mathbb{R}^n$$

Hence, for our particular h_0, we also know that

$$\mu\langle h_0,h_0\rangle - \langle Ah_0,h_0\rangle \geq 0$$

Since $\langle Ax_1-\mu x_1,h_0\rangle < 0$ and since we assume that $\langle Ax_1-\mu x_1,h_0\rangle < \mu\langle h_0,h_0\rangle - \langle Ah_0,h_0\rangle$, we note that we can choose $\alpha > 0$ so that

$$-2\langle Ax_1-\mu x_1,h_0\rangle > \alpha[\mu\langle h_0,h_0\rangle - \langle Ah_0,h_0\rangle]$$

$\Rightarrow$

$$-2\alpha\langle Ax_1-\mu x_1,h_0\rangle > \alpha^2[\mu\langle h_0,h_0\rangle - \langle Ah_0,h_0\rangle]$$

$$2\langle Ax_1-\mu x_1,-\alpha h_0\rangle > \mu\langle \alpha h_0,\alpha h_0\rangle - \langle A(\alpha h_0),(\alpha h_0)\rangle$$

$$2\langle Ax_1-\mu x_1,-\alpha h_0\rangle + \langle A(\alpha h_0),(\alpha h_0)\rangle - \mu\langle \alpha h_0,\alpha h_0\rangle > 0$$

$$2\langle Ax_1-\mu x_1,-\alpha h_0\rangle + \langle A(-\alpha h_0),(-\alpha h_0)\rangle - \mu\langle -\alpha h_0,-\alpha h_0\rangle > 0$$

Hence, we have produced $\hat{h} = -\alpha h_0 \in \mathbb{R}^n$ so that

$$2\langle Ax_1-\mu x_1,\hat{h}\rangle + \langle A\hat{h},\hat{h}\rangle - \mu\langle \hat{h},\hat{h}\rangle > 0$$

Hence, this contradicts our given inequality. //

[AND BY CONTRADICTION ARGUMENT, WE'VE PROVEN OUR CLAIM!]