Section 1.6 Applications of E-values / E-vectors

Section 1.6.1 Exponentiation of Matrices - e^{At}

Given $\frac{du}{dt} = au$, $u(0) = u_0$, where a is constant, $u(t)$ is a real-valued function.

The soln is $u(t) = u_0 e^{at}$ (soph. ODE class)

Recall the motivating example for Section 1.2 for a 2x2 system example.

Given $A \in \mathbb{R}^{n \times n}$, $x = x(t)$ is a vector in $\mathbb{R}^n$, then what can we say about the system below?

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0 \qquad (1.15)$$

- If A is diagonalizable, then we can answer this question! So, if $\exists$ a similarity transformation so that

$$A = T^{-1} \Lambda T$$

with $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, (See Thms 1.2 - 1.5 for details of these matrices) then we can write

$$TAT^{-1} = \Lambda$$

and

$$\begin{aligned} T \frac{dx}{dt} &= TAx \\ &= TA(T^{-1}T)x \\ &= [TAT^{-1}]Tx \\ &= \Lambda \underbrace{Tx}_{\text{Define } y = Tx} \end{aligned}$$

Note that since T is a matrix of constants, then

$$\frac{dy}{dt} = \frac{d}{dt}(Tx) = T\frac{dx}{dt}$$

And our ODE system becomes

$$\frac{dy}{dt} = \Lambda y \quad , \text{ with } \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

$$\Rightarrow \quad y(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}}_{= e^{\Lambda t}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

matrix.

Then $y(0) = Tx(0)$

$$\underbrace{e^{\Lambda(0)}}_{\text{Identity matrix}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = Tx_0 \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = Tx_0$$

$$\Rightarrow \quad y(t) = e^{\Lambda t} Tx_0$$

$$\Rightarrow \quad Tx(t) = e^{\Lambda t} Tx_0$$

$$x(t) = \underbrace{T^{-1} e^{\Lambda t} T}\, x_0$$

define matrix $\rightarrow$ $e^{At} = T^{-1} e^{\Lambda t} T$

$x(t) = e^{At} x_0$, where $e^{At} = T^{-1} e^{\Lambda t} T$