LU Decomposition

$$A = LU \quad , A \in \mathbb{R}^{n\times m}, L \in \mathbb{R}^{n\times n}, U \in \mathbb{R}^{n\times m}$$

- $U \in \mathbb{R}^{n\times m}$ will have a block of zeros if $m \neq n$.
- L, U are found by performing Gauss Elimination on A and keeping a record of the details!
- G.E. can be represented by a series of matrix multiplications on A.

Begin with:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

Using a_{11} as the pivot element, we "zero out" all the entries under a_{11} by multiplying by the matrix

$$M_1 = \begin{bmatrix} 1 & & & & 0 \\ -\sigma_2^{(1)} & 1 & & & \\ -\sigma_3^{(1)} & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ -\sigma_n^{(1)} & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad \sigma_i^{(1)} = \frac{a_{i1}}{a_{11}}, \quad i = 2, 3, \ldots n$$

- Then $M_1 A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \\ 0 & a_{n2}^{(1)} & & a_{nn}^{(1)} \end{bmatrix}$

Similarly,

$$M_2 = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & 0 & \\ \vdots & -\sigma_3^{(2)} & 1 & & & \\ \vdots & \vdots & 0 & \ddots & & \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & -\sigma_n^{(2)} & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad \underbrace{\sigma_i^{(2)} = \frac{a_{i2}^{(1)}}{a_{22}^{(1)}}, \quad i = 3, 4, \ldots n}$$

Note: I'm being sloppy here. these σ_i's are NOT the SAME as the σ_i's in the previous step of G.E.

So,

$$M_2 M_1 A \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & & a_{2n}^{(1)} \\ \vdots & 0 & a_{33}^{(2)} & & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & a_{n3}^{(2)} & & a_{nn}^{(2)} \end{bmatrix}$$

Applying the process $n-1$ times (AND yes, I'm tacitly assuming that we don't encounter any zero pivot elements and that the process is "completeable" on my A matrix), we get

$$M_{n-1} M_{n-2} \cdots M_2 M_1 A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & 0 & \ddots & \\ 0 & 0 & & a_{nn}^{(n-1)} \end{bmatrix}}_{\text{Upper } \Delta\text{'lar matrix } U}$$

$$\Rightarrow \quad A = M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1} U$$

- Note: Each M_k is invertible with M_k^{-1} having the very convenient form

$$M_k^{-1} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ 0 & & 1 & & & \\ \vdots & & \sigma_{k+1}^{(k)} & \ddots & & \\ \vdots & & \sigma_{k+2} & & \ddots & \\ 0 & & \sigma_n^{(k)} & 0 & & 1 \end{bmatrix}$$

- Furthermore, $M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1} = L$ and has the form

$$L = \begin{bmatrix} 1 & & & & \\ \sigma_2^{(1)} & 1 & & 0 & \\ \vdots & \sigma_3^{(2)} & 1 & & \\ \vdots & \vdots & & \ddots & \\ \sigma_n^{(1)} & \sigma_n^{(2)} & \cdots & \sigma_n^{(n-1)} & 1 \end{bmatrix}$$

← Note: L has Lin. Ind. Columns

- Observe: (A) L is Lower Δ'lar
  (B) Ones on diagonal of L
  (C) G.E. multipliers for each step sit under the diagonal elements.

- An advantage of having the LU-decomposition is that if we are given a Linear system
$$Ax = b$$
to solve, the LU-decomp. can make this process reasonably painless - by first using forward substitution + then using backward substitution on the appropriate Linear systems →

$$Ax = b$$
$$LUx = b \qquad \text{, define } y = Ux$$
$$Ly = b$$

1. Solve $Ly = b$ using forward substitution
2. Solve $Ux = y$ using back substitution

- This yields the soln x.

- Note: This can be very beneficial if we are working with an algorithm that requires repeated linear system solves where the matrix A doesn't change and we only need to use a new b-vector during each iteration of the algorithm.

- Pseudoinverse of A using LU:

Thm 1.11
If $A = LU$, then $A' = U^*(UU^*)^{-1}(L^*L)^{-1}L^*$

Why? Verify that $x = A'b$ satisfies $A^*Ax = A^*b$

$$A^*Ax = (LU)^* LU\left[U^*(UU^*)^{-1}(L^*L)^{-1}L^*\right] b$$

$$= \left[U^*L^*L\underbrace{(UU^*)(UU^*)^{-1}}_{=I}(L^*L)^{-1}L^*\right] b$$

$$= U^*(L^*L)(L^*L)^{-1}L^*b \qquad \longrightarrow$$

$$
\begin{aligned}
A^*Ax &= U^*L^*b \\
&= (LU)^*b \\
&= A^*b
\end{aligned}
$$

So, for x defined by $x = A'b$, it satisfies the normal eqns, and now we need to check that our x is orthogonal to all $w \in N(A)$.

If $w \in N(A)$, then $Aw = 0 \Rightarrow LUw = 0$.
Since L has linearly independent cols, then L^{-1} exists, and this implies that it must be the case that

$$Uw = 0.$$

Then

$$
\begin{aligned}
\langle w, A'b \rangle &= \langle w, U^*(UU^*)^{-1}(L^*L)^{-1}L^*b \rangle \\
&= \langle Uw, (UU^*)^{-1}(L^*L)^{-1}L^*Lb \rangle \\
&= \langle 0, (UU^*)^{-1}(L^*L)^{-1}L^*Lb \rangle \\
&= 0
\end{aligned}
$$

since $Uw = 0$
i.e. $w \in N(U)$

Hence, $x = A'b$ is orthogonal to all $w \in N(A)$.
And the A' defined above satisfies the requirements of Defn. 1.13.

Ex: $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ ROW REDUCE $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

So, $A = LU \Rightarrow \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

<u>Note</u>: To compute the Pseudoinverse of A, we first eliminate the zero rows of U and Corresponding Cols. of L to form the L, U that contain only the Lin. Ind. Cols + Rows.

So, Begin with $L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$.

<u>New L and U</u>:

$$L = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$$

Then the Pseudoinverse is given by

$$A' = U^*(UU^*)^{-1}(L^*L)^{-1}L^*$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \left[\begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right]^{-1} \left[\begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \left(\frac{1}{56}\right) \cdot \left(\frac{5}{4}\right)^{-1} \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{70} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{70} \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{bmatrix}$$

← See pg. 31 for G.E. version of this computation.