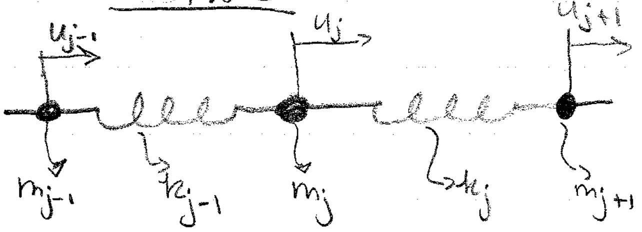
Toy Example / Application: (pg. 22-24)

A Lattice of (one-dimensional) balls connected by Linear Springs:

Focus on one ball + its neighbors on the left & right – Picture:



Let t = time

$u_j(t)$ = displacement (from equilibrium position) of the j^{th} ball. Motion to the right of equilibrium position is considered $u_j(t) > 0$

m_j = mass of j^{th} ball

k_j = Spring constant associated with j^{th} spring according to diagram above. Assume Hooke's Law + "Restoring force of j^{th} spring is Linearly proportional to its displacement from equilibrium with constant of proportionality $= k_j > 0$"

- Assume that only the nearest neighbor balls exert any force on the j^{th} ball.

- Assume Newton's 2nd Law governs the motion ($F=ma$)

<u>EQNs of Motion:</u>

$$m_j (u_j)'' = k_j (u_{j+1} - u_j) + k_{j-1}(u_{j-1} - u_j) \quad , j = 1, 2, \ldots n$$

$$\Rightarrow \quad u_j'' = \frac{k_j}{m_j} u_{j-1} - \frac{(k_{j-1} + k_j)}{m_j} u_j + \frac{k_j}{m_j} u_{j+1}$$

Define

$$\vec{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

We write the 2nd-order System as $\frac{d^2 \vec{u}}{dt^2} = Au$ (1.8)

Where

$$A = \begin{bmatrix} -\frac{(k_0+k_1)}{m_1} & k_1/m_1 & 0 & \cdots & \cdots & 0 \\ \frac{k_1}{m_1} & -\frac{(k_1+k_2)}{m_2} & \frac{k_2}{m_2} & \ddots & & \vdots \\ 0 & \ddots & & & \ddots & 0 \\ \vdots & & \ddots & & & \frac{k_n}{m_n} \\ 0 & & & 0 & \frac{k_n}{m_n} & -\frac{(k_{n-1}+k_n)}{m_n} \end{bmatrix}$$

A is tridiagonal. The e-values of the Differential operator are the e-values of the matrix A.

- E-values of A correspond to Natural Frequencies at which each of the springs vibrates. E-vectors ϕ determine the shape of the vibrating mode.

Look for $u(t) = e^{i\omega t}\phi \Rightarrow u'(t) = i\omega e^{i\omega t}\phi$, $u'' = -\omega^2 e^{i\omega t}\phi$

Plugging into Eqn (1.8), we have

$$-\omega^2 e^{i\omega t}\phi = A e^{i\omega t}\phi \Rightarrow \underbrace{A\phi = -\omega^2\phi}_{\text{e-pair expression!}}$$

- Consider Case where $m_j = m \ \forall i = 1, 2, \ldots n$, then A is symmetric

$$A = \frac{1}{m}\begin{bmatrix} -(k_0+k_1) & k_1 & & O \\ k_1 & -(k_1+k_2) & \ddots & \\ & \ddots & \ddots & k_{n-1} \\ O & & k_{n-1} & -(k_{n-1}+k_n) \end{bmatrix}$$

We can use MiniMax Principle to investigate e-values. We examine the values

$q(x) = \langle Ax, x \rangle$

$$= \left\langle \frac{1}{m}\begin{bmatrix} -(k_0+k_1)x_1 + k_1x_2 \\ k_1x_1 \quad -(k_1+k_2)x_2 \quad k_2x_3 \\ \vdots \\ k_{n-1}x_{n-1} - (k_{n-1}+k_n)x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right\rangle$$

$$= \frac{1}{m}\left[\sum_{j=1}^{n} -(k_{j-1}+k_j)x_j^2 + 2\sum_{j=1}^{n-1} k_j x_j x_{j+1}\right]$$

$$= \frac{1}{m}\left[\sum_{j=1}^{n} -k_{j-1}x_j^2 - k_j x_j^2 + 2\sum_{j=1}^{n-1} k_j x_j x_{j+1}\right]$$

$$= \frac{1}{m}(-1)\left[\sum_{j=1}^{n} k_{j-1}x_j^2 + \sum_{j=1}^{n} k_j x_j^2 - \sum_{j=1}^{n-1} 2k_j x_j x_{j+1}\right]$$

$$= -\frac{1}{m}\left[\sum_{j=0}^{n-1} k_j x_{j+1}^2 + \sum_{j=1}^{n} k_j x_j^2 - \sum_{j=1}^{n-1} 2k_j x_j x_{j+1}\right]$$

$$= -\frac{1}{m}\left[k_0x_1^2 + k_nx_n^2 + \sum_{j=1}^{n-1} k_j\left[x_j^2 - 2x_jx_{j+1} + x_{j+1}^2\right]\right]$$

So, $$q(x) = -\frac{1}{m}\left[k_0x_1^2 + k_n^2x_n^2 + \sum_{j=1}^{n-1} k_j(x_j - x_{j+1})^2\right]$$

Since $k_j > 0$ for all $j = 1, 2, \ldots n$, then for any nonzero vector $\vec{x} = [x_1, x_2, \ldots, x_n]^T$, we see that

$$q(x) = -\frac{1}{m}\left[k_0 x_1^2 + k_n^2 x_n^2 + \sum_{j=1}^{n-1} k_j (x_j - x_{j+1})^2 \right]^2 < 0$$

- Hence, the matrix A is Negative definite

- If we increase k_j for any j, then the quantity $q(x)$ is made more negative, so the values of $q(x)$ are decreased. Hence, the e-values of A are decreased.

- If m is increased, the e-values remain negative, but they are made less negative. Hence, they are smaller in absolute value. And since the e-values represent the natural frequencies of the vibrations, then increasing m means decreasing the natural frequencies of vibration.

- Increasing any spring constant k_j makes the e-values more negative (and larger in absolute value). Hence, this corresponds to increasing the natural frequencies of the vibration.

- These natural frequencies are called Resonant frequencies. That is, if the system is exposed to an external force vibrating at the same frequency, then the system resonates!

Basic Example of Resonance: (1 mass, 1 spring)

(*) $u'' + 9u = 2\cos(3t)$, $u(0)=0$, $u'(0)=0$

Soln:

Homog. EQN: $u'' + 9u = 0$

Char. EQN: $r^2 + 9 = 0$ $r = \pm 3i$

$$u_h(t) = C_1\cos(3t) + C_2\sin(3t)$$

Particular Soln:

MUDC's Attempt: $u_p(t) = t[A\cos(3t) + B\sin(3t)]$

$$u_p(t) = t[A\cos(3t) + B\sin(3t)]$$

$$u_p'(t) = t[-3A\sin(3t) + 3B\cos(3t)] + [A\cos(3t) + B\sin(3t)]$$

$$u_p''(t) = t[-9A\cos(3t) - 9B\sin(3t)] + [-3A\sin(3t) + 3B\cos(3t)] + [-3A\sin(3t) + 3B\cos(3t)]$$

$$= t[-9A\cos(3t) - 9B\sin(3t)] + [-6A\sin(3t) + 6B\cos(3t)]$$

Plugging into (*):

$$t[-9A\cos(3t) - 9B\sin(3t)] + [-6A\sin 3t + 6B\cos(3t)] + 9t[A\cos(3t) + B\sin(3t)] = 2\cos(3t)$$

$\Rightarrow$

$$-6A\sin(3t) + 6B\cos(3t) = 2\cos(3t)$$

$\Rightarrow$

$$A = 0, \ B = \tfrac{1}{3} \Rightarrow u_p(t) = \tfrac{1}{3}t\sin(3t)$$

$$u(t) = C_1\cos(3t) + C_2\sin(3t) + \tfrac{1}{3}t\sin(3t)$$

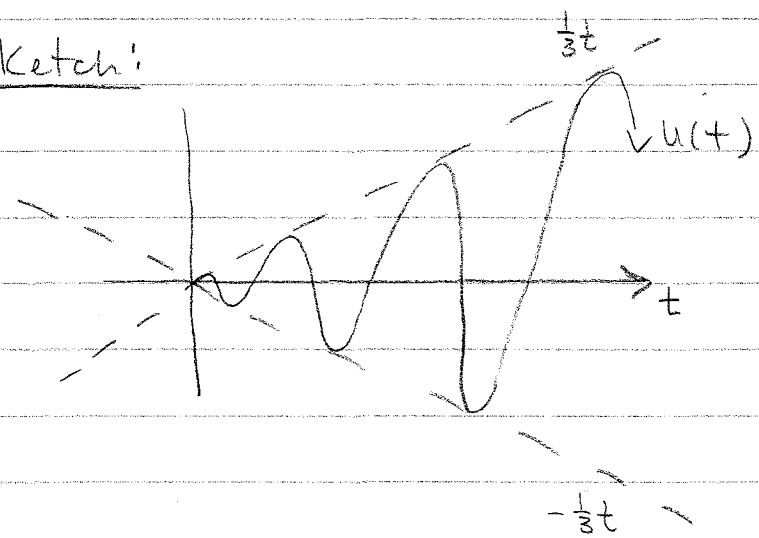
I.C.s $u(0) = 0 \Rightarrow C_1 = 0$

$u'(0) = 0 \Rightarrow -3C_1\sin(0) + 3C_2\cos(0) + \tfrac{1}{3}\sin(0) - (0)\cos(0) = 0$

$$C_2 = 0$$

So, $u(t) = \frac{1}{3}t\sin(3t)$

Sketch:



- In theory, the displacement of the spring oscillates with increasing amplitude (without bound).

- Mathematically, this is caused by the fact that frequency of the external force, $2\cos(3t)$, is exactley the same as that of the natural frequency of the system - identified by

$$u_h(t) = c_1\cos(3t) + c_2\sin(3t)$$

That is, both angular frequencies are 3!

- If we considered

$$u'' + 9u = 2\cos(2.5t), \; u(0)=0, \; u'(0)=0,$$

then the soln would exhibit bounded oscillations as $t \to +\infty$.