SVD & E-values/E-vectors of A^TA & AA^T

There is a nice relationship between the SVD for $A \in \mathbb{R}^{n \times m}$ and E-values/E-vectors of the matrices $A^TA \in \mathbb{R}^{m \times m}$ and $AA^T \in \mathbb{R}^{n \times n}$

Recall the Picture: (Still assuming $\text{rank}(A) = r$)

$$
\begin{array}{lcl cl}
 & A & & A^T & \\
v_1 & \xrightarrow{\sigma_1} & u_1 & \xrightarrow{\sigma_1} & v_1 \\
v_2 & \xrightarrow{\sigma_2} & u_2 & \xrightarrow{\sigma_2} & v_2 \\
\vdots & & \vdots & & \vdots \\
v_r & \xrightarrow{\sigma_r} & u_r & \xrightarrow{\sigma_r} & v_r \\
\left.\begin{array}{l} v_{r+1} \\ v_{r+2} \\ \vdots \\ v_m \end{array}\right\} & \Rightarrow 0 & \left.\begin{array}{l} u_{r+1} \\ u_{r+2} \\ \vdots \\ \vdots \\ u_n \end{array}\right\} & \rightarrow 0 & \begin{array}{l} v_{r+1} \\ \vdots \\ v_m \end{array}
\end{array}
$$

For the SVD of A:

$$Av_i = \begin{cases} \sigma_i u_i, & i = 1, 2, \ldots r \\ 0, & i = r+1, \ldots m \end{cases} \quad \text{and} \quad A^T u_i = \begin{cases} \sigma_i v_i, & i = 1, 2, \ldots r \\ 0, & i = r+1, r+2, \ldots n \end{cases}$$

So for $i = 1, 2, \ldots r$

$$A^TAv_i = A^T(\sigma_i u_i) = \sigma_i A^T u_i = \sigma_i(\sigma_i v_i) = \sigma_i^2 v_i$$

Hence, for $i = 1, 2, \ldots r$, the squares of the singular values, σ_i^2, are e-values of A^TA with corresponding e-vectors v_i.

Similarly,

$$AA^T u_i = A(\sigma_i v_i) = \sigma_i A v_i = \sigma_i(\sigma_i u_i) = \sigma_i^2 u_i$$

So, for $i = 1, 2, \dots r$, the squares of the singular values, σ_i^2, are e-values of AA^T with corresponding e-vectors u_i.

In addition,

- $N(A^TA) = N(A)$ and $N(AA^T) = N(A^T)$
- $\text{rank}(A^TA) = \text{rank}(A) = \text{rank}(A^T) = \text{rank}(AA^T) = r$
- So, the SVD of A identifies the range spaces $R(AA^T)$ and $R(A^TA)$ along with $N(AA^T)$ and $N(A^TA)$.
- This also shows us that the e-values of AA^T and A^TA, are all non-negative. In particular, each of AA^T and A^TA have exactly r positive e-values. The multiplicity of the e-value $\lambda = 0$ depends on the value of r, related to m and n.

$A^TA \in \mathbb{R}^{m \times m}$, so A^TA has a zero eigenvalue of multiplicity $m-r$.

$AA^T \in \mathbb{R}^{n \times n}$, so AA^T has a zero eigenvalue of multiplicity $n-r$.

Two Disclaimers:

1. σ_i's are uniquely determined, but just as eigenvectors of length 1 are not unique because we can always multiply the vector by -1 and get another e-vector of length 1, we can do the same for right + left singular vectors v_i's and u_i's.

2. I've tacitly assumed that I can always find a linearly independent set of right + left singular vectors (ie. e-vectors), but if we have repeated eigenvalues | singular values, then this can be messy!

Finally, an EXAMPLE!

- There are many ways to compute an SVD - here will compute AA^T just to connect it with previous discussion.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$A^TA \in \mathbb{R}^{3\times3}$ and $AA^T \in \mathbb{R}^{2\times2}$, so we use AA^T

$$AA^T = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}, \quad p(\lambda) = (5-\lambda)(8-\lambda) - 4 = 0$$

$$\lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = \frac{13 \pm \sqrt{169 - 4(36)}}{2} = \frac{13 \pm 5}{2}$$

E-values of AA^T are : $\lambda_1 = 9$, $\lambda_2 = 4$

Singular Values of A are: $\sigma_1 = 3$, $\sigma_2 = 2$
(Note: $\text{rank}(A) = 2$ so that $r = 2$ here)

Left Singular Vectors of A: these are e-vectors of AA^T

$\lambda_1 = 9$ Let $u_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ and find u_1 so that $\|u_1\| = 1$ and
$AA^T u_1 = 9u_1$ (ie. $[AA^T - 9I]u_1 = \vec{0}$)

$$\left(\begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \right) u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -4x + 2y &= 0 \\ 2x - y &= 0 \end{aligned}\right\}$$ these eqns should give the same info

$\Rightarrow \quad y = 2x$

Choose $u_1 = \dfrac{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 4$ Let $u_2 = \begin{bmatrix} x \\ y \end{bmatrix}$ and find u_2 so that $\|u_2\| = 1$
and $[AA^T - 4I]u_2 = \vec{0}$

$$\Rightarrow \left(\begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x + 2y &= 0 \\ 2x + 4y &= 0 \end{aligned}\right\} \Rightarrow x = -2y \text{ or } y = -\tfrac{1}{2}x$$

Choose $u_2 = \dfrac{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\|} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Observe that $u_1 \perp u_2$ - according to theory, they must be!

Right Singular Vectors of A:

- We could compute A^TA and find its e-vectors to get the Right Singular vectors v_1, v_2, v_3. But, an easier, more efficient way to do this is to use the relationship stated in the SVD theorem that is $\quad A^Tu_i = \sigma_i v_i \ , \ i=1,\dots r$

i.e. $\quad v_i = \frac{1}{\sigma_i} A^T u_i \quad$ for $i=1,2$ in this case

$$v_1 = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}\left(\frac{1}{\sqrt{5}}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad v_2 = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}\left(\frac{1}{\sqrt{5}}\right)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$v_1 = \frac{1}{3\sqrt{5}}\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} \qquad v_2 = \frac{1}{2\sqrt{5}}\begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

To obtain v_3, we can use the relationship from the SVD Thm that $Av_3 = 0$, or we could apply Gram-Schmidt using v_1, v_2 - and then normalize to get v_3:

Gram-Schmidt

Choose this vector so that it's Lin. Ind. with v_1, v_2

$$\phi_1 = v_1$$

$$\phi_2 = v_2$$

$$\phi_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\langle v_1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\rangle}{\|v_1\|} v_1 - \frac{\langle v_2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\rangle}{\|v_2\|} v_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{3\sqrt{5}}(11)}{1}\cdot\left(\frac{1}{3\sqrt{5}}\right)\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} - \frac{\frac{1}{\sqrt{5}}(1)}{1}\cdot\left(\frac{1}{\sqrt{5}}\right)\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{11}{45}\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

So, $v_3 = \phi_3 / \|\phi_3\|$ $\rightarrow$

Normalizing $v_3 = \dfrac{\frac{1}{9}\begin{bmatrix}-2\\1\\2\end{bmatrix}}{\left\|\frac{1}{9}\begin{bmatrix}-2\\1\\2\end{bmatrix}\right\|} = \dfrac{\begin{bmatrix}-2\\1\\2\end{bmatrix}}{3} = \frac{1}{3}\begin{bmatrix}-2\\1\\2\end{bmatrix}$

Putting it all together:

$$U = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}, \quad \Sigma = \begin{bmatrix}3 & 2 & 0\\0 & 0 & 0\end{bmatrix}, \quad V = \frac{1}{3\sqrt{5}}\begin{bmatrix}5 & 0 & -2\sqrt{5}\\2 & 6 & \sqrt{5}\\4 & -3 & 2\sqrt{5}\end{bmatrix}$$

- Check that $A = U\Sigma V^T$
- Note that $A^T = V\Sigma U^T$
- Also observe that A^TA and AA^T have e-values that are non-negative. This agrees with other items from linear algebra theory.

1. A^TA and AA^T are positive semidefinite. That is, $x^T(A^TA)x \geq 0 \ \forall x \neq 0$ and $x^T(AA^T)x \geq 0 \ \forall x \neq 0$

Why? $x^TA^TAx = (Ax)^T(Ax) = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0 \ \forall x \neq 0.$

(Similarly for AA^T)

2. All e-values of AA^T & A^TA are non-negative (See your homework for this one)