Section 2.1 (Cont'd)

In applied math, it is often the case that an "exact" soln to a given problem is not known, not useful for analysis or otherwise unreliable because of various simplifying assumptions made during the soln process. Bottom Line - It's messy, icky and prone to "grey area". For such reasons, iterative processes are often employed in order to generate a sequence of approximations to a soln. So, sequences & their convergence is important for us.

Defn 2.1

A sequence $\{x_n\}$ is S (a vector space) is said to have a limit x in S if there is $x \in S$ such that for any $\varepsilon > 0$, there is an integer N so that for all $n \geq N$,

$$\|x_n - x\| < \varepsilon$$

- Here, the integer N is usually dependent upon the size of ε.

- We say that $x_n \to x$ in S, or we use the notation $\lim_{n \to \infty} x_n = x$

We say that "x_n converges to x in S".

Ex: Let $x_n = 1 - \frac{1}{n}$, then $x_n \to 1$ in $\mathbb{R}$.

Ex: Let $\{f_n\}$ be a sequence in $C[0,1]$ defined by

$$f_n(x) = e^{\frac{1}{n}x}, \quad n = 1, 2, 3, \ldots$$

We use the usual "sup"-norm to impose on $C[0,1]$.

Claim: $f_n \to 1$ in $C[0,1]$ as $n \to \infty$.

pf: Let $\varepsilon > 0$. We want to find N so that for all $n \geq N$,

$$\| f_n - 1 \| < \varepsilon$$

First, $\| f_n - 1 \| = \max\limits_{x \in [0,1]} | f_n(x) - 1 | = \max\limits_{x \in [0,1]} | e^{\frac{1}{n}x} - 1 |$

We note that $e^{\frac{1}{n}x}$ is a monotonically increasing function for $x \in [0,1]$, and $e^{\frac{1}{n}x} - 1 \geq 0$ for all $x \in [0,1]$. Hence

$$\| f_n - 1 \| = \max_{x \in [0,1]} | e^{\frac{1}{n}x} - 1 | = \max_{x \in [0,1]} e^{\frac{1}{n}x} - 1 = e^{\frac{1}{n}} - 1$$

So, I want to choose N so that $n \geq N$ implies that $e^{\frac{1}{n}} - 1 < \varepsilon$, but this is equivalent to

$$e^{\frac{1}{n}} < 1 + \varepsilon$$

$$\frac{1}{n} < \ln(1+\varepsilon)$$

$$n > \frac{1}{\ln(1+\varepsilon)}$$

If we choose an integer $N > \frac{1}{\ln(1+\varepsilon)}$, then

$$n \geq N \Rightarrow n > \frac{1}{\ln(1+\varepsilon)} \Rightarrow e^{\frac{1}{n}} - 1 < \varepsilon$$

$$\Rightarrow \| f_n - 1 \| < \varepsilon$$

Hence $f_n \to 1$ in $C[0,1]$ with the 'sup-norm.

A concept of closeness that doesn't involve a limiting vector.

<u>Defn. 2.2</u>
A sequence $\{x_n\}$ is S is called a Cauchy Sequence if for any $\varepsilon > 0$ there is an integer N so that for every $m, n \geq N$, $\|x_m - x_n\| < \varepsilon$.

- Note: the value of N depends on ε

* A convergent sequence is a Cauchy Sequence
<u>Why?</u>
If $x_n \to x$ in S, then for any $\varepsilon > 0$, we can find an integer N so that if $n \geq N$, then

$$\|x_n - x\| < \varepsilon/2$$

So, if we consider, $\|x_m - x_n\|$, we can use the Δ inequality to obtain that for $m, n \geq N$,

$$\begin{aligned}
\|x_m - x_n\| &= \|x_m - x + x - x_n\| \\
&= \|(x_m - x) + (x - x_n)\| \\
&\leq \|x_m - x\| + \|x - x_n\| \quad \text{by } \Delta \text{ ineq.} \\
&= \|x_m - x\| + \|x_n - x\| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}$$

Hence, if $m, n \geq N$, then $\|x_m - x_n\| < \varepsilon$, and the sequence $\{x_n\}$ is also a Cauchy sequence.

- A Cauchy sequence is not necessarily a convergent sequence.

Ex: Let $S = C[0,1]$ with the 1-norm given by
$$\|f\| = \int_0^1 |f(x)|\,dx$$

One can define a sequence $\{f_n(x)\}$ given by

$$f_n(x) = \begin{cases} n, & \text{if } x \in [0, \frac{1}{n^2}] \\ x^{-1/2}, & \text{if } x \in [\frac{1}{n^2}, 1] \end{cases}$$

It turns out that this sequence is Cauchy in S, but $f_n(x)$ does not converge to a limit f that is contained in S.

Defn 2.3

A normed linear (vector) space S is Complete if every Cauchy Sequence in S is $\underbrace{\text{convergent in } S}$.

Means Two Things
1. Converges to a limit $x_n \to x$
2. AND $x \in S$ also.

A "BANACH SPACE" is a complete Normed Linear Space.

* $\mathbb{Q}$ = set of all rational #s is NOT COMPLETE
* $\mathbb{R}$ = set of all real #s is COMPLETE Normed linear space.
* Not all vector spaces are Complete
* Concept of Completeness is highly dependent on the norm that one is imposing on the space.

$C[0,1]$ with $\|f\| = \max_{x\in[0,1]} |f(x)|$ is <u>Complete!</u>

$C[0,1]$ with $\|f\| = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$ is <u>NOT complete!</u>

<u>Discussion</u>

- Spse $\{f_n(t)\}$ is a Cauchy sequence in $C[0,1]$ <u>with</u> respect to the "uniform" norm.

- We show that we can define a Limiting function $f(t)$ with $f(t) \in C[0,1]$ so that $f_n(t) \to f(t)$ w.r.t. uniform norm.

For a fixed point $\hat{t} \in [0,1]$, the sequence $\{f_n(\hat{t})\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, then the Cauchy sequence $\{f_n(\hat{t})\}$ converges to a value, call it $f(\hat{t})$. Note that we can do this for each $\hat{t} \in [0,1]$ and construct the function $f(t)$ so that

$$f_n(t) \to f(t) \quad \text{for each } t \in [0,1].$$

We call $f(t)$ the "ptwise Limit" of the sequence $\{f_n(t)\}$

[Now, we want to show that the function $f(t)$ is a cont. function $\forall t \in [0,1]$ so that $f \in C[0,1]$.]

We are given that $\{f_n(t)\}$ is a Cauchy sequence in $C[0,1]$ w.r.t. the "uniform" norm. Then given any $\varepsilon > 0$ we can find $N \in \mathbb{Z}^+$ so that for $n, m \geq N$, we have

$$\|f_n - f_m\| = \max_{t\in[0,1]} |f_n(t) - f_m(t)| < \varepsilon/3 \qquad (*)$$

just stating continuity defn. here

[We also know that for each n, $f_n(t)$ is a continuous function for all $t \in [0,1]$. This means that for any $\varepsilon > 0$ there exists a $\delta > 0$ so that if $|h| \le \delta$, then

$$|f_n(t+h) - f_n(t)| < \varepsilon$$]

And I can achieve this for any n that I choose.

Now, choose a fixed integer $k \geq N$ (the N from previous page)

Then by continuity, $\exists\ \delta > 0$ so that if $|h| \le \delta$,

then $\quad |f_k(t+h) - f_k(t)| < \varepsilon/3$

Putting all of this together, if $|h| \le \delta$, & remember $k \geq N$, we have

$$|f(t) - f(t+h)| = |f(t) - f_k(t) + f_k(t) - f_k(t+h) + f_k(t+h) - f(t+h)|$$

$$\le |f(t) - f_k(t)| + |f_k(t) - f_k(t+h)| + |f_k(t+h) - f(t+h)|$$

$$< \underbrace{\varepsilon/3}_{\substack{\text{pt. wise limit} \\ \text{of } f_k(t)}} + \underbrace{\varepsilon/3}_{\substack{\text{by continuity} \\ \text{of } f_k(t)}} + \underbrace{\varepsilon/3}_{\substack{\text{by pt. wise limit} \\ \text{of } f_k(t+h)}}$$

$$= \varepsilon$$

Hence, for any $\varepsilon > 0$, $\exists\ \delta > 0$ so that if $|h| \le \delta$, then

$$|f(t) - f(t+h)| < \varepsilon,$$

and by definition $f \in C[0,1]$.

<u>Note</u>: What really makes this work is the "uniform" nature of the "uniform" norm. So, when we look at the expression in (*) on pg. 5, we see then $\|f_n - f_m\| < \varepsilon/3$ means that the functions

are "close" for all $t \in [0,1]$ and that the positive integer N does not depend on t in any way. The "uniform" norm takes away that issue.

So, given $\varepsilon > 0$, $\exists N$ so that for any $n, m \geq N$

$$\| f_n - f_m \| < \varepsilon$$

means $\quad | f_n(t) - f_m(t) | < \varepsilon \quad \underline{\underline{\forall\, t \in [0,1]}}$.

And then

$$\lim_{m \to \infty} | f_n(t) - f_m(t) | = | f_n(t) - \lim_{m \to \infty} f_m(t) |$$

$$= | f_n(t) - f(t) |$$

$$< \varepsilon \qquad \text{for all } t \in [0,1].$$

That is, for $n \geq N$,

$$\| f_n - f \| = \max_{t \in [0,1]} | f_n(t) - f(t) | < \varepsilon.$$

So, $f_n \to f$ in $C[0,1]$.

The following example comes from "Introductory Functional Analysis with Applications" by Erwin Kreyszig Copyright 1978, Wiley & Sons.

→ I borrowed Ken's copy. There is a reprint in the Wiley Classics series that is paperback + somewhat newer.

$C[0,1]$ with the L^p-norms $(1 \le p < \infty)$ is NOT COMPLETE!
We use $C[0,1]$ with the L^1-norm here. That is,
if $f \in C[0,1]$, we define

$$\|f\| = \int_0^1 |f(t)|\,dt$$

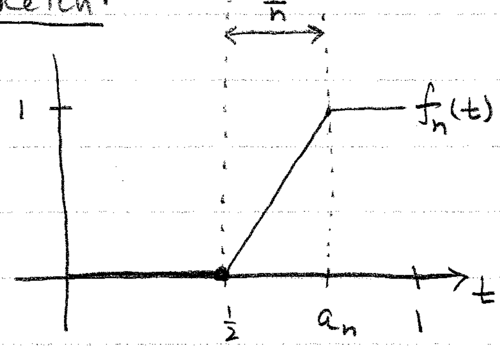
Consider the sequence $\{f_n(t)\}$ defined by

$$a_n = \frac{1}{2} + \frac{1}{n}, \quad n = 1, 2, \ldots$$

and

$$f_n(t) = \begin{cases} 0, & \text{if } t \in [0, 1/2] \\ n(t - \frac{1}{2}), & t \in [\frac{1}{2}, a_n] \\ 1, & \text{if } t \in [a_n, 1] \end{cases}$$

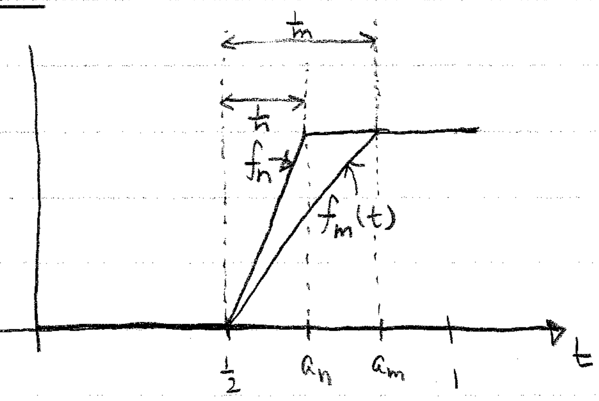
<u>Sketch:</u>



Here's one element of the sequence $\{f_n(t)\}$

Since, we will be examining $\|f_n - f_m\| = \int_0^1 |f_n(t) - f_m(t)|\,dt$,
we look at a sketch of two elements of the sequence
superimposed on the same axes.

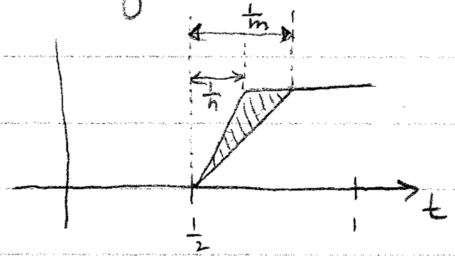
<u>Sketch:</u>



Claim: the functions $\{f_n\}$ form a Cauchy sequence in $C[0,1]$ with L^1-norm.

pf: Examine $\|f_n - f_m\| = \int_0^1 |f_n(t) - f_m(t)|\, dt$

= Area of Triangle shown below



So, $\|f_n - f_m\| = \left|\frac{1}{2}(1)\left(\frac{1}{n}\right) - \frac{1}{2}(1)\left(\frac{1}{m}\right)\right| = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right|$

Then if we are given any $\varepsilon > 0$, we can choose a positive integer N so that $N > \frac{1}{\varepsilon}$. And if $m, n \geq N$, then we have

$$\begin{aligned} \|f_n - f_m\| &= \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \\ &\leq \frac{1}{2}\left(\left|\frac{1}{n}\right| + \left|\frac{1}{m}\right|\right) \\ &= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m}\right) \\ &< \frac{1}{2}(\varepsilon + \varepsilon) \\ &= \varepsilon \end{aligned}$$

$\left\{\begin{array}{l} \text{if } m \geq N, \text{ then } \frac{1}{m} \leq \frac{1}{N} < \varepsilon \\ \text{if } n \geq N, \text{ then } \frac{1}{n} \leq \frac{1}{N} < \varepsilon \end{array}\right.$

Hence, for $m, n \geq N > \frac{1}{\varepsilon}$, we have

$$\|f_n - f_m\| < \varepsilon,$$

and by defn., $\{f_n\}$ is a Cauchy Sequence in $C[0,1]$ w.r.t. the L^1-norm.

// End of Claim

* The next step is to show that this sequence does not converge to a limit $f(t)$ in $C[0,1]$. →

Let $f \in C[0,1]$, then

$$\|f_n(t) - f(t)\| = \int_0^1 |f_n(t) - f(t)|\,dt$$

$$= \int_0^{\frac{1}{2}} |f_n(t) - f(t)|\,dt + \int_{\frac{1}{2}}^{a_n} |f_n(t) - f(t)|\,dt + \int_{a_n}^{1} |f_n(t) - f(t)|\,dt$$

$$= \underbrace{\int_0^{\frac{1}{2}} |f(t)|\,dt}_{\text{Ⓐ non-negative}} + \underbrace{\int_{\frac{1}{2}}^{a_n} |f_n(t) - f(t)|\,dt}_{\text{non-negative}} + \underbrace{\int_{a_n}^{1} |1 - f(t)|\,dt}_{\text{Ⓑ non-negative}}$$

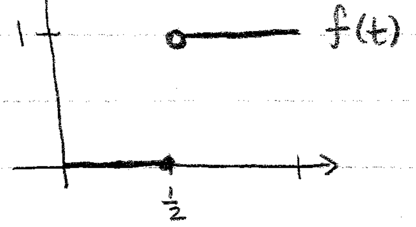
So, if $\|f_n - f\| \to 0$ as $n \to \infty$, then each term of the integral must $\to 0$ as $n \to \infty$. Since $f(t)$ is a continuous function, then we must have

$f(t) = 0$ for all $t \in [0, \frac{1}{2}]$ (so that term Ⓐ is zero)

AND

$f(t) = 1$ for all $t \in (\frac{1}{2}, 1)$ (so that term Ⓑ is zero as $n \to \infty$ i.e. as $a_n \to \frac{1}{2}$)

So, the sketch of $f(t)$ is as you might imagine:



This is impossible for a continuous function $f \in C[0,1]$. Hence, f_n does not converge to a limit that is actually contained in the space $C[0,1]$. So, $C[0,1]$ is NOT complete when it's equipped with the L^1-norm (OR any of the L^p-norms)

<u>Remark</u>: If all this seems bizarre, an important comment to remember is that the concepts of "Cauchy Sequence" and "Convergent Sequence" are both norm-dependent concepts. So, one thing to note is that the example we just presented that is Cauchy when we use the L^1-norm is <u>NOT</u> a Cauchy sequence when we use the "uniform" norm. So, realize that the "uniform" norm is a <u>very</u> <u>restrictive</u> measure of distance that requires a certain amount of uniformity over the entire interval of interest.