From Brenner & Scott

$$\left[\begin{array}{c} -u''(x) = f(x), \quad x \in (0,1) \\ \underbrace{u(0) = 0}_{\text{Dirichlet}}, \quad \underbrace{u'(1) = 0}_{\text{Neumann}} \end{array}\right] \quad (P1)$$

$$-u'' \cdot \phi = f\phi \qquad \text{for all } \phi \in V = H^1_L = \{v : v' \in L^2(0,1) \ \& \ v(0) = 0\}$$

$$-\int_0^1 u''(x)\phi(x)\,dx = \int_0^1 f(x)\phi(x)\,dx \qquad \text{"} \qquad \text{"}$$

$$-\left[\phi(x)u'(x)\Big|_0^1 - \int_0^1 u'(x)\phi'(x)\,dx\right] = \int_0^1 f(x)\phi(x)\,dx$$

$$-\left(\underbrace{\phi(0)}_{\text{"essential"}=0}u'(0) - \phi(1)\underbrace{u'(1)}_{=0 \text{ "natural"}}\right) + \int_0^1 u'(x)\phi'(x)\,dx = \int_0^1 f(x)\phi(x)\,dx$$

$$\int_0^1 u'(x)\phi'(x)\,dx = \int_0^1 f(x)\phi(x)\,dx \qquad \forall\, \phi \in V$$

Define the bilinear form

$$a(u,v) = \int_0^1 u'(x)v'(x)\,dx \quad \text{for all } u, v \in V$$

and the L^2-inner product

$$\langle f, v\rangle = \int_0^1 f(x)\phi(x)\,dx \quad \text{for all } v \in V$$

Then the variational problem is to:

$$\left[\begin{array}{l} \text{Find } u \in V \text{ so that} \\ \qquad a(u,\phi) = \langle f, \phi\rangle \quad \forall\, \phi \in V \end{array}\right] \quad (P2)$$

(*) Under some assumptions, (P1) + (P2) are equivalent

Why do we care? When we use finite elements, we approximate the solution to (P2), so we want to know that we are also approximating the soln to (P1) when we discretize.

Clearly if u solves (P1), then u solves (P2).

Theorem 0.1.4
Spse $f \in C^0[0,1]$ and $u \in C^2[0,1]$ which satisfy (P2). Then u also solves (P1).

proof: Let $\phi \in V \cap C^1[0,1]$. Then the weak form gives us

$$\langle f, \phi \rangle = a(u,\phi) = \int_0^1 u'(x)\phi'(x)\,dx$$

$$= u'(x)\phi(x)\Big|_0^1 - \int_0^1 u''(x)\phi(x)\,dx$$

$$= \int (-u''(x))\phi(x)\,dx + u'(1)\phi(1)$$

$$= \langle -u'', \phi \rangle + u'(1)\phi(1)$$

Then $\forall\ \phi \in V \cap C^1[0,1]$ with the condition that $\phi(1)=0$, we have

$$0 = \langle \underbrace{f+u''}_{w}, \phi \rangle = \int_0^1 (f(x)+u''(x))\phi(x)\,dx$$

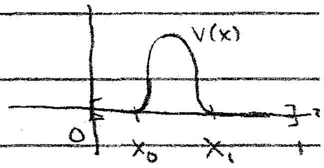
Let $w = f(x) + u''(x) \in C^0[0,1]$.

Ⓐ If $w \equiv 0$, then $-u''(x) = f(x)\ \forall\ x \in [0,1]$

Ⓑ If $w \not\equiv 0$, then there exists $x_0, x_1 \in [0,1]$ with $x_0 < x_1$ so that $w(x)$ is of one sign on the interval $[x_0, x_1]$

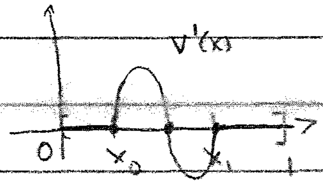
Choose the function

$$v(x) = \begin{cases} (x-x_0)^2(x-x_1)^2, & x \in [x_0,x_1] \\ 0, & \text{otherwise} \end{cases}$$



Note:

$$v'(x) = \begin{cases} \underbrace{2(x-x_0)^2(x-x_1) + 2(x-x_0)(x-x_1)^2}_{=2(x-x_0)(x-x_1)(2x-x_0-x_1)}, & x \in [x_0,x_1] \\ 0, & \text{otherwise} \end{cases}$$



Therefore, $v \in V \cap C^1[0,1]$

Furthermore $\langle w, v\rangle = \int_0^1 w(x)v(x)\,dx = \int_{x_0}^{x_1} w(x)v(x)\,dx \neq 0$
since $v>0$ on (x_0,x_1) and w is either positive or negative on $[x_0,x_1]$. This is a contradiction, so we must have that condition (A) holds;

ie. $-u''(x) = f(x)\ \forall x\in[0,1]$

Since $u\in V$, then $u(0)=0$, but we need the Neumann Boundary Condition:

Since $\langle f,\phi\rangle = \langle -u'',\phi\rangle + u'(1)\phi(1)$ holds $\forall \phi\in V\cap C^1[0,1]$, then it must be true for $\phi(x)=x$. But this gives

$$\langle f, x\rangle = \langle -u'', x\rangle + u'(1)(1)$$

$$\int_0^1 \underbrace{[f(x)-(-u''(x))]}_{=0}\,x\,dx = u'(1)$$

$$0 = u'(1)$$

Therefore, u satisfies the following

$$-u'' = f,\ x\in(0,1)$$
$$u(0)=0,\ u'(1)=0$$

ie., u solves (P1). //

(*) The strong solution (ie. soln to classical ODE) always satisfies the weak formulation. Occasionally, the classical ODE will not have a solution in the classical sense so that only the variational formulation has a solution. If the weak form has a soln and if the ODE has a strong solution, then the those two solutions must be the same.

Boundary Conditions:

$u(0)=0 \Rightarrow$ "essential" OR Dirichlet

$u'(1)=0 \Rightarrow$ "natural" OR Neumann

Ritz-Galerkin Approximation

Let $S \subset V$ be a finite dimensional subspace. Consider approximating the variation problem (P2) by:

(P3) Find $u_S \in S$ so that $a(u_S, \phi) = \langle f, \phi \rangle \; \forall \phi \in S$

Hence, we use $u_S \approx u$ the soln to (P2) & (P1).

Theorem (0.2.2)

If $f \in L^2(0,1)$, then (P3) has a unique solution

pf: Choose a Basis for S given by $B = \{\phi_i : i=1,2,\dots,n\}$

Let $u_S = \sum_{j=1}^{n} u_j \phi_j$; Let $K_{ij} = a(\phi_j, \phi_i)$ for $i,j = 1,2,\dots n$

form the matrix $K \in \mathbb{R}^{n\times n}$, and let $b_i = \langle f, \phi_i \rangle$, $i=1,2,\dots n$

form the vector $\vec{b} \in \mathbb{R}^{n\times 1}$. Then if we construct the vector

$\vec{u} = [u_1 \; u_2 \dots u_n]^T$, we have the Linear system

$$K\vec{u} = \vec{b}$$

which is equivalent to solving (P3). Note K is Symmetric since

$$K_{ij} = a(\phi_j, \phi_i) = \int_0^1 \phi_j'(x)\phi_i'(x)\,dx = \int_0^1 \phi_i'(x)\phi_j'(x)\,dx = K_{ji}$$

Furthermore, K is Positive Definite (Assign for HW #4)

$$\vec{v}^T K \vec{v} = v^T \begin{bmatrix} \sum_{j=1}^{n} K_{1j} v_j \\ \sum_{j=1}^{n} K_{2j} v_j \\ \vdots \\ \sum_{j=1} K_{nj} v_j \end{bmatrix} = \sum_{j=1}^{n} K_{1j} v_1 v_j + \sum_{j=1}^{n} K_{2j} v_2 v_j + \dots + \sum_{j=1}^{n} K_{nj} v_n v_j$$

$\therefore \vec{v}^T K \vec{v} = \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij} v_i v_j$

But if we use the elements of the vector $\vec{v}$ to form the function $v(x) \in S$ given by $v(x) = \sum_{j=1}^{n} v_j \phi_j$, then

$$0 \le a(v,v) = \int_0^1 v' v' \, dx$$

$$= \int_0^1 \Big(\sum_{j=1}^{n} v_j \phi_j'\Big)\Big(\sum_{i=1}^{n} v_i \phi_i'\Big) dx$$

$$= \int_0^1 \sum_{j=1}^{n} \sum_{i=1}^{n} v_i v_j \, \phi_i' \phi_j' \, dx$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} v_i v_j \int_0^1 \phi_j' \phi_i' \, dx$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} K_{ij} v_i v_j = \vec{v}^T K \vec{v}$$

Note: there's a little more to this argument; in particular, $a(v,v) = 0 \Rightarrow v = 0$. And some of the deeper details of that argument require some work in Sobolev spaces. (See Brenner & Scott for details)

Hence $\vec{v}^T K \vec{v} > 0 \ \forall \vec{v} \neq 0$, and K is SPD.
Since K is SPD, K is invertible, and the soln of (P3) exists & is unique. //

Error Estimates

The approximation u_s exhibits an orthogonality property

u satisfies:

$$a(u,v) = \langle f, v \rangle \quad \forall \, v \in V$$

u_s satisfies:

$$a(u_s, v) = \langle f, v \rangle \quad \forall \, v \in S \subset V$$

$\hookrightarrow$ a finite dimension subspace

$\therefore u_s$ satisfies

$$(*) \quad (0.3.1) \qquad a(u - u_s, v) = 0 \quad \forall \, v \in S$$

This says $u - u_s \perp v \ \forall v \in S$ where "orthogonality" is w.r.t. the inner-product $a(u,v)$ $\forall u, v \in S$

(*) guides our error estimates:

Define the energy norm

$$\|v\|_E = \sqrt{a(v,v)} = \left(\int_0^1 [v'(x)]^2 dx\right)^{1/2}$$

Using the Schwarz' Inequality

$$|a(u,v)| \le \|u\|_E \|v\|_E$$

This implies that for any $v \in S$

$$\begin{aligned}
\|u-u_S\|_E^2 &= a(u-u_S, u-u_S) \\
&= a(u-u_S, u-v+v-u_S) \\
&= a(u-u_S, u-v) + \underbrace{a(u-u_S, \underbrace{v-u_S}_{\in S})}_{=0 \text{ by } (*)} \\
&= a(u-u_S, u-v) \\
&\le \|u-u_S\|_E \|u-v\|_E .
\end{aligned}$$

$\therefore \quad \|u-u_S\|_E \le \|u-v\|_E \quad \forall\, v \in S$

(as long as $u \notin S$)

Note: If $u \in S$, then $u - u_S = 0$, and if $\|u-u_S\|_E = 0$ then the above inequality is trivial.

Therefore,

$$\|u-u_S\|_E \le \inf\{\|u-v\|_E : v \in S\} \text{ and } \le \|u-u_S\|_E$$

And we have already shown that u_S exists, and $u_S \in S$. Therefore, we have shown that

(0.3.3) Thm $\quad \|u-u_S\|_E = \min\{\|u-v\|_E : v \in S\}$

Thm 0.3.3 • BASIC Error Estimate for FEM.
Can be used with particular choices for S in order to derive more concrete estimates.
• the Error is optimal in the energy norm

Now, we consider the error in the L^2-norm
Define $\|v\|_2 = \langle v, v\rangle^{1/2} = \left(\int_0^1 [v(x)]^2 dx\right)^{1/2} \quad \forall\, v \in L^2(0,1)$.
How does the $\|u-u_S\|_2$ relate to $\|u-u_S\|_E$?
With appropriate assumptions, $\|\cdot\|_2 << \|\cdot\|_E$
To estimate this, we use a "duality" argument

Let w solve
$$-w'' = u - u_S, \quad x \in (0,1) \qquad (\text{Note: } u-u_S \text{ is our error!})$$
$$w(0)=0, \quad w'(1)=0$$

Assumption:
$$\inf_{v\in S} \|w-v\|_E \le \epsilon w'' \qquad (0.3.4)$$

Theorem (0.3.5)
If w satisfies Assumption (0.3.4), then
$$\|u-u_S\|_2 \le \epsilon \|u-u_S\|_E \le \epsilon^2 \|u''\|_2 = \epsilon^2 \|f\|_2.$$

$\Big\{$ * $\|u-u_S\|_E \le \epsilon \|f\|_2$ is $O(\epsilon)$ approx.
But $\|u-u_S\|_2 \le \epsilon^2 \|f\|_2$ is $O(\epsilon^2)$ approx. $\Big]$

see proof on next page ⇒

$$\|u-u_S\|_2^2 = \langle u-u_S, u-u_S\rangle = \int_0^1 [u-u_S]^2\,dx$$
$$= \langle u-u_S, -w''\rangle$$
$$= -\int_0^1 (u-u_S)\,w''\,dx$$
$$= -\Big[(u-u_S)w'\Big|_0^1 - \int_0^1 (u'-u_S')\,w'\,dx\Big]$$
$(u-u_S)(0)=0$, $w'(1)=0$
$$= \int_0^1 (u'-u_S')\,w'\,dx$$

$$= a(u-u_S, w)$$
$$= a(u-u_S, w) - a(u-u_S, v) \quad \text{for } v\in S \text{ by } (0.3.1)$$
$$= a(u-u_S, w-v)$$

Then $\|u-u_S\|_2^2 = a(u-u_S, w-v)$
$$= |a(u-u_S, w-v)|$$
$$\le \|u-u_S\|_E \cdot \|w-v\|_E \quad \forall v\in S$$

$$\therefore \quad \|u-u_S\|_2 \le \frac{\|u-u_S\|_E \cdot \|w-v\|_E}{\|u-u_S\|_2}$$
$$= \frac{\|u-u_S\|_E\,\|w-v\|_E}{\|-w''\|_2} = \frac{\|u-u_S\|_E \cdot \|w-v\|_E}{\|w''\|_2} \quad \forall v\in S$$

$$\therefore \quad \|u-u_S\|_2 \le \frac{\|u-u_S\|_E}{\|w''\|_2} \cdot \underbrace{\inf_{v\in S} \|w-v\|_E}_{\le \epsilon \|w''\|_2 \text{ by } (0.3.4)}$$

$$\therefore \quad \|u-u_S\|_2 \le \epsilon \|u-u_S\|_E$$

And if we assume that u also satisfies (0.3.4), then
$$\|u-u_S\|_2 \le \epsilon \|u-u_S\|_E \le \epsilon \inf_{v\in S}\|u-v\| \le \epsilon^2 \|u''\| = \epsilon^2 \|f\|. //$$

Theorem (0.3.5) relies on the (reasonable) assumption that we can find $v \in S$ so that v is very close to w in the energy norm. "How close" may depend on our choice of S - our finite dimensional subspace.

Now, we consider a family of spaces S for which we can get v that is very close. That is, ϵ can be made arbitrarily small.

this means

* If we choose our finite element basis carefully, we can get convergence in $\|\cdot\|_2$ or in the $\|\cdot\|_E$.

0.4 Piecewise Polynomial Space - F.E.M.

Partition the interval $[0,1]$ by

$0 = x_0 < x_1 < x_2 < \dots x_N < x_{N+1} = 1$, and

let S be the linear space of functions ϕ such that

i.) $\phi \in C^0[0,1]$

ii.) $\phi|_{[x_{i-1}, x_i]}$ is a linear polynomial $\forall i = 1, 2, \dots, N+1$

iii.) $\phi(0) = 0$

Then $S \subseteq V$, and we define ϕ_i by the requirement that $\phi_i(x_j) = \delta_{ij}$ (See diagram of hat functions from earlier in your notes)

See next page

- The set $\{\Phi_i\}_{i=1}^{N+1}$ is a NODAL BASIS for S.
- The pts $\{x_i\}$ are called the NODES.
- If V is a function in S, then the pts. $\{V(x_i)\}_{i=0}^{N+1}$ are the NODAL VALUES of the function V.

That is,

$$V(x) = \sum_{i=1}^{N+1} \underbrace{V(x_i)}\Phi_i$$

↳ these are our coefficients

Defn: Given $v \in C^0[0,1]$, the interpolant $V_I \in S$ of v is the piecewise linear function which interpolates the function v at the nodes $x_1, x_2, \ldots, x_{N+1}$. And V_I is given by

$$V_I = \sum_{i=1}^{N+1} V(x_i)\Phi_i(x)$$

(Note: By the defn of S, we require $V_I(x_0)=0$, so we don't interpolate v at the pt. x_0 unless $V(x_0)=0$.)

Lemma (0.4.1)
$\{\Phi_i\}_{i=1}^{N+1}$ is a basis for S.

pf: Linear Independence of $\{\Phi_i\}$
Spse $c_1\Phi_1 + c_2\Phi_2 + \cdots + c_{N+1}\Phi_{N+1} = 0$ for all $x \in [0,1]$
In particular, for $j=1,2,\ldots,N+1$

$$\sum_{i=1}^{N+1} c_i\Phi_i(x_j) = 0 \Rightarrow c_j\Phi_j(x_j) = 0 \text{ by construction of } \Phi_i\text{'s.}$$
$$\Rightarrow c_j(1) = 0$$
$$\Rightarrow c_j = 0$$

Hence $c_j = 0 \ \forall\, j=1,2,\ldots,N+1$, and the functions $\{\Phi_i\}$ form a linearly independent set.

$\{\phi_i\}$ spans S

The set $\{\phi_i\}$ spans S if $v \in S \Rightarrow v = v_I$

Claim: $v \in S \Rightarrow v = v_I$

pf: Define $z(x) = v(x) - v_I(x)$. Since $v \in S$ & $v_I \in S$, then each of these is linear on $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, N$.

Hence $z \in S$ also, and by defn.

$$v(x_i) = v_I(x_i) \quad \text{for } i = 1, 2, \ldots, N+1$$

Then

$$z(x_i) = 0 \quad \text{for } i = 1, 2, \ldots, N+1$$

Also, $z(x_0) = v(x_0) = v_I(x_0) = 0$

Therefore, $z(x)$ is linear on $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, N$ with $z(x_i) = z(x_{i+1}) = 0$. Hence, $z(x)$ must be the zero function on $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots N$

Therefore $z(x) \equiv 0$, and

$$v(x) \equiv v_I(x).$$

Hence, if $v \in S$, then its representation for the basis $\{\phi_i\}$ is given by the interpolant.

And $\{\phi_i\}$ spans S. //

Hence, $\{\phi_i\}$ is a basis for S. //

Question:

Given a function $u \in V$, if we construct the interpolant u_I, how much error is there in this approximation?

Theorem (0.4.5)

Let $h = \max_{1 \le i \le N+1} \{x_i - x_{i-1}\}$. Then

$$\|u - u_I\|_E \le C h \|u''\|_2$$

for all $u \in V$, where C is independent of h and u.

pf: Note $\|u-u_I\|_E^2 = \int_0^1 [u'(x) - u_I'(x)]^2 dx = \sum_{j=1}^{N+1} \int_{x_{j-1}}^{x_j} [u'(x) - u_I'(x)]^2 dx$

and

$$\|u''\|_2^2 = \int_0^1 [u''(x)]^2 dx = \sum_{j=1}^{N+1} \int_{x_{j-1}}^{x_j} [u''(x)]^2 dx$$

We prove the inequality piecewise and then sum. That is, we want to show that

$$\int_{x_{j-1}}^{x_j} [u'(x) - u_I'(x)]^2 dx \le c\,(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} [u''(x)]^2 dx \qquad (*)$$

Begin with the error function $e(x) = u(x) - u_I(x)$

Since u_I is a linear polynomial on $[x_{j-1}, x_j]$, then the expression in $(*)$ is equivalent to

$$\int_{x_{j-1}}^{x_j} [e'(x)]^2 dx \le c(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} [e''(x)]^2 dx.$$

Note: We can use an affine transformation of $[x_{j-1}, x_j]$ to $[0,1]$ by $T : [x_{j-1}, x_j] \to [0,1]$ by

$$T(x) = \tilde{x} = \frac{1}{x_j - x_{j-1}} \cdot [x - x_{j-1}]$$

Furthermore, $T^{-1} : [0,1] \to [x_{j-1}, x_j]$

$$T^{-1}(\tilde{x}) = x = x_{j-1} + \tilde{x}(x_j - x_{j-1})$$

Likewise, the transformed error function is

$$\tilde{e}(\tilde{x}) = e(x_{j-1} + \tilde{x}(x_j - x_{j-1}))$$

defined on $[0,1]$.

Then under this mapping, the above inequality is equivalent to showing

$$\int_0^1 [\tilde{e}'(\tilde{x})]^2 d\tilde{x} \le c \int_0^1 [\tilde{e}''(\tilde{x})]^2 d\tilde{x} \qquad (**)$$

We have reduced a mesh-dependent estimate (*) to a mesh independent estimate in (**). This type of reduction argument is called a <u>homogeneity argument</u> OR a <u>scaling argument</u>. Hence, we only need to verify (**)

Let $z = \tilde{e}$ and use x instead of $\tilde{x}$ as our variable.

<u>Note</u>: $z(0) = \tilde{e}(0) = e(x_{j-1}) = u(x_{j-1}) - u_I(x_{j-1}) = 0$ &

$z(1) = \tilde{e}(1) = e(x_j) = u(x_j) - u_I(x_j) = 0$

Since u_I interpolates u at x_{j-1} & x_j. Then by Rolle's Thm, there exists $\xi \in (0,1) \ni$

$$z'(\xi) = 0.$$

Then we can write

$$z'(y) = \int_\xi^y z''(x)\,dx$$

&

$$|z'(y)| = \left|\int_\xi^y z''(x)\,dx\right|$$

$$= \left|\int_\xi^y 1 \cdot z''(x)\,dx\right|$$

$$\le \left|\int_\xi^y 1^2 dx\right|^{1/2} \left|\int_\xi^y [z''(x)]^2 dx\right|^{1/2} \text{ by C-Schwarz}$$

$$= |y-\xi|^{1/2} \left|\int_\xi^y [z''(x)]^2 dx\right|^{1/2}$$

$$\le |y-\xi|^{1/2} \left(\int_0^1 [z''(x)]^2 dx\right)^{1/2}$$

$\therefore \quad (z'(y))^2 \le |y-\xi| \cdot \int_0^1 [z''(x)]^2 dx$

$$\int_0^1 [z'(y)]^2 dy \le \int_0^1 [z''(x)]^2 dx \cdot \int_0^1 |y-\xi|\, dy$$

$$\le \int_0^1 [z''(x)]^2 dx \cdot \underbrace{\sup_{0 \le \xi \le 1} \int_0^1 |y-\xi|\, dy}_{=\frac{1}{2}}$$

See HW #5

$\therefore \; C = \frac{1}{2}$, and

$$\int_0^1 [z'(y)]^2 dy \le C \int_0^1 [z''(x)]^2 dx$$

$$\therefore \quad \int_0^1 [z'(x)]^2 dx \le C \int_0^1 [z''(x)]^2 dx$$

and this is equivalent to $(*)$.

$$\|u-u_I\|_E^2 = \int_0^1 [u'(x)-u_I'(x)]^2 dx = \sum_{j=1}^{N+1} \int_{x_{j-1}}^{x_j} [e'(x)]^2 dx$$

$$\le \sum_{j=1}^{N+1} c\,(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} [e''(x)]^2 dx$$

$$\le c h^2 \sum_{j=1}^{N+1} \int_{x_{j-1}}^{x_j} [e''(x)]^2 dx$$

$$= c h^2 \int_0^1 [e''(x)]^2 dx$$

$$= c h^2 \int_0^1 [u''(x)]^2 dx \quad \text{since } u_I''(x) = 0$$

$$= c h^2 \|u''(x)\|_2^2$$

$\therefore \quad \|u-u_I\|_E \le \underbrace{\sqrt{c}}_{C=\sqrt{c}}\, h \,\|u''(x)\|_2$, and the proof is finished. //

Corollary (0.4.7)

$$\|u-u_s\| + Ch\,\|u-u_s\|_E \le 2(Ch)^2\|u''\|$$

pf: Thm (0.4.5) shows that Assumption (0.3.4) holds with $\epsilon = Ch$. Therefore,

$$\|u-u_s\| \le \epsilon^2\|u''\| \quad \text{by Thm (0.3.5)}$$
$$= (Ch)^2\|u''\|$$

and

$$\epsilon\,\|u-u_s\|_E \le \epsilon^2\|u''\| \quad \text{by Thm (0.3.5)}$$
$$\therefore \quad Ch\,\|u-u_s\|_E \le (Ch)^2\|u''\|$$

Hence, $\|u-u_s\| + Ch\,\|u-u_s\|_E \le 2(Ch)^2\|u''\|$ //

Read the Section 0.5 on "Relationship to Difference Methods"

Computer Implementation

For 1D problems: $u(x)$, where $x\in\mathbb{R}^1$, then the programming is very straightforward. It can really be done without the need for a lot of generalizations, but you should still try to organize your code in a modular way.

I. Initialize parameters – $p(x)$, $q(x)$, $f(x)$, Bdry Cond's, Type of Basis Fctns.

II. Geometry Module

III. Assembly of the system matrix. – includes integration

IV. Solution of Linear System – pick your favorite

V. Recovery of the solution for error analysis + plotting purposes.