The Finite Element Method for B.V.P.s

① Serious Mathematical theory underlying the method. Requires some functional analysis at its highest level.

② We will begin with B.V.P.s in 1 spatial dimension

③ As before, we have a grid
Now - we have basis functions too, and we basically project the true solution to the D.E. onto a particular space of functions. We get to choose our basis fctns for the fctn. space.

Example:

$$(*) \qquad -(p(x)u'(x))' + q(x)u(x) = g(x), \quad x \in (0,1)$$
$$u(0) = a, \quad u(1) = b.$$
$$p > 0 \text{ and } q \geq 0 \quad \forall x \in [0,1]$$

We develop the weak formulation of the eqn (*)

$$\int -(pu')'\phi + qu\phi \, dx = \int g\phi \, dx \quad \forall \phi \in W = H_0^1(0,1)$$

$\hookrightarrow$ "test" functions

$$-\int (pu')'\phi \, dx + \int qu\phi \, dx = \int g\phi \, dx$$

$$-\left[pu'\phi\Big|_0^1 - \int pu'\phi' \, dx\right] + \int qu\phi \, dx = \int g\phi \, dx \quad \forall \phi \in H_0^1(0,1)$$

$$\underbrace{-pu'\phi\Big|_0^1}_{=0 \text{ since } \phi(0)=\phi(1)=0} + \int pu'\phi' \, dx + \int qu\phi \, dx = \int g\phi \, dx$$

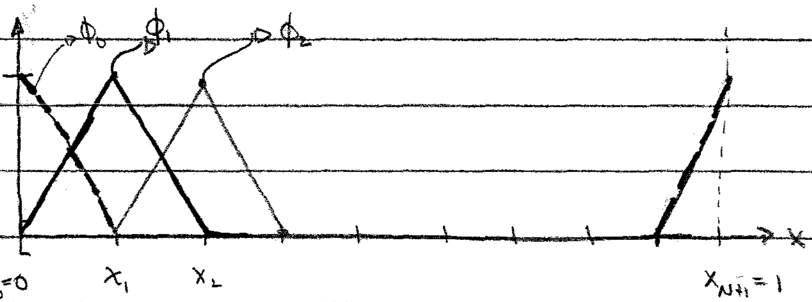
$$\therefore (**) \quad \int pu'\phi' \, dx + \int qu\phi \, dx = \int g\phi \, dx \quad \forall \phi \in H_0^1(0,1)$$

$\hookrightarrow$ the "weak formulation"

Discretization:

1. Grid pts: Choose $N \in \mathbb{Z}^+$ & define $h = \frac{1}{N+1}$

Define $x_i = ih$ for $i = 0, 1, 2, \ldots$



2. Basis functions : Piecewise Polynomials

$\boxed{\text{A) Linear}}$
B) Quadratic
C) Cubic

(Generally, choosing the polynomial depends on the smoothness of the function to be computed)

The piecewise linear basis functions are called the "hat" functions, ϕ_i, defined so that

i. ϕ_i is piecewise linear

ii. $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ for all $j \neq i$

(i.e. $\phi_i(x_j) = \delta_{ij}$ ← the kronecker delta)

Leads to:

$$\phi_i(x) = \begin{cases} \frac{1}{h}(x - x_{i-1}), & \text{if } x \in [x_{i-1}, x_i] \\ \frac{1}{h}(x_{i+1} - x), & \text{if } x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{cases}$$

3. Construct our approximation $\hat{u}(x) \approx u(x)$ by

$$\hat{u}(x) = \sum_{j=0}^{N+1} \alpha_j h_j(x)$$

Our job is to compute the values of α_j for $j = 0, 1, 2, \ldots N+1$

Note: It should be the case that $\hat{u}(0) = u(0) = a$ and $\hat{u}(1) = u(1) = b$ since we are given those values in the problem statement. This leads to

$$a = \hat{u}(0) = \sum_{j=0}^{N+1} \alpha_j h_j(0) = \alpha_0 h_0(0) = \alpha_0(1) = \alpha_0$$

and

$$b = \hat{u}(1) = \sum_{j=0}^{N+1} \alpha_j h_j(1) = \alpha_{N+1} h_{N+1}(1) = \alpha_{N+1}(1) = \alpha_{N+1}$$

So, we get these two for free in the problem, and now we only need to compute α_j for $j = 1, 2, \ldots, N$.

4. Hence, we return to the weak form:
We want to find $u \in H_0^1(0,1)$ which satisfies (**) $\forall\, \phi \in H_0^1(0,1)$.
Strategically choose a few representative functions for $\phi \rightarrow$ namely ϕ_i, where $i = 1, 2, \ldots N$. This will result in a linear system $A\vec{x} = \vec{b}$. These eqns are described by

Weak Form Eqn (**)

$$\int p(x)[\hat{u}(x)]'\phi'\,dx + \int q(x)[\hat{u}(x)]\phi\,dx = \int g(x)\phi(x)\,dx \quad \forall\, \phi \in H_0^1(0,1)$$

Introduce approx. with

$$\int p(x)\Big[\underbrace{\sum_{j=0}^{N+1}\alpha_j\phi_j}_{\hat{u}}\Big]'\phi'\,dx + \int q(x)\Big[\underbrace{\sum_{j=0}^{N+1}\alpha_j\phi_j}_{\hat{u}}\Big]\phi\,dx = \int g(x)\phi(x)\,dx$$

We require that this eqn is satisfied for $\phi = \phi_i$ for $i = 1, 2, \ldots, N$.

$$\int p(x)\Big[\sum_{j=0}^{N+1}\alpha_j\phi_j'\Big]\phi_i'\,dx + \int q(x)\Big[\sum_{j=0}^{N+1}\alpha_j\phi_j\Big]\phi_i\,dx = \int g(x)\phi_i\,dx$$

for $i = 1, 2, \ldots, N$.

The Nature of the Basis Functions leads to a special structure if we re-write our previous eqn as:

$$\sum_{j=0}^{N+1} \alpha_j \int p(x)\,\phi_j'(x)\,\phi_i'(x)\,dx + \sum_{j=0}^{N+1} \alpha_j \int q(x)\,\phi_j(x)\,\phi_i(x)\,dx = \int g(x)\,\phi_i(x)\,dx$$

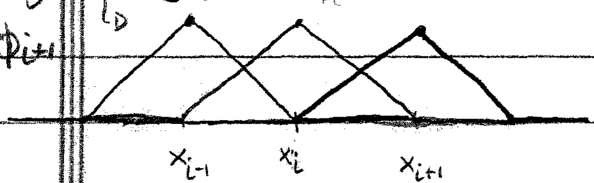
for $i=1,2,\ldots,N$

Observe:

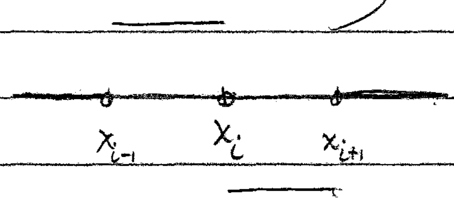
$\int p(x)\,\phi_j'(x)\,\phi_i'(x)\,dx \neq 0$ only for $|i-j| \leq 1$

$\int q(x)\,\phi_j(x)\,\phi_i(x)\,dx \neq 0$ only for $|i-j| \leq 1$

Sketch of $\phi_{i-1}, \phi_i, \phi_{i+1}$



Sketch of $\phi_i'(x)$



Our Boundary Condition terms get thrown to the R.H.S.

$i=1$

$$\underbrace{\alpha_0}_{=a} \int p(x)\phi_0'(x)\phi_1'(x)dx + \alpha_1 \int p(x)\phi_1'(x)\phi_1'(x)dx + \alpha_2 \int p(x)\phi_2'(x)\phi_1'(x)dx$$

$$+ \alpha_0 \int q(x)\phi_0(x)\phi_1(x)dx + \alpha_1 \int q(x)\phi_1(x)\phi_1(x)dx + \alpha_2 \int q(x)\phi_2(x)\phi_1(x)dx$$

$$= \int g(x)\phi_1(x)dx$$

$$\alpha_1 \int p(x)\phi_1'(x)\phi_1'(x)dx + \alpha_2 \int p(x)\phi_2'(x)\phi_1'(x)dx + \alpha_1 \int q(x)\phi_1(x)\phi_1(x)dx + \alpha_2 \int q(x)\phi_2(x)\phi_1(x)dx$$

$$= \int g(x)\phi_1(x)dx - a\left(\int p(x)\phi_0'(x)\phi_1'(x)dx + \int q(x)\phi_0(x)\phi_1(x)dx\right)$$

$i=2$

$$\alpha_1 \int p(x)\phi_1'(x)\phi_2'(x)dx + \alpha_2 \int p(x)\phi_2'(x)\phi_2'(x)dx + \alpha_3 \int p(x)\phi_3'(x)\phi_2'(x)dx$$

$$+ \alpha_1 \int q(x)\phi_1(x)\phi_2(x)dx + \alpha_2 \int q(x)\phi_2(x)\phi_2(x)dx + \alpha_3 \int q(x)\phi_3(x)\phi_2(x)dx$$

$$= \int g(x)\phi_2(x)dx$$

$\underline{N-1}$

$$\alpha_{N-2}\int p(x)\phi'_{N-2}\phi'_{N-1}\,dx + \alpha_{N-1}\int p(x)\phi'_{N-1}\phi'_{N-1}\,dx + \alpha_N\int p(x)\phi'_N\phi'_{N-1}\,dx$$
$$+ \alpha_{N-2}\int q(x)\phi_{N-2}\phi_{N-1}\,dx + \alpha_{N-1}\int q(x)\phi_{N-1}\phi_{N-1}\,dx + \alpha_N\int q(x)\phi_N\phi_{N-1}\,dx$$
$$= \int g(x)\phi_{N-1}\,dx$$

$\underline{i=N}$

$$\alpha_{N-1}\int p(x)\phi'_{N-1}\phi'_N\,dx + \alpha_N\int p(x)\phi'_N\phi'_N\,dx + \overbrace{\alpha_{N+1}}^{=b}\int p(x)\phi'_{N+1}\phi'_N\,dx$$
$$+ \alpha_{N-1}\int q(x)\phi_{N-1}\phi_N\,dx + \alpha_N\int q(x)\phi_N\phi_N\,dx + \underbrace{\alpha_{N+1}}_{=b}\int q(x)\phi_{N+1}\phi_N\,dx$$
$$= \int g(x)\phi_N\,dx$$

$$\alpha_{N-1}\int p(x)\phi'_{N-1}\phi'_N\,dx + \alpha_N\int p(x)\phi'_N\phi'_N\,dx + \alpha_{N-1}\int q(x)\phi_{N-1}\phi_N\,dx + \alpha_N\int q(x)\phi_N\phi_N\,dx$$
$$= \int g(x)\phi_N\,dx - b\left(\int p(x)\phi'_{N+1}\phi'_N\,dx + \int q(x)\phi_{N+1}\phi_N\,dx\right)$$

If we define $\vec{\alpha} = [\alpha_1\ \alpha_2\ \dots\ \alpha_N]^T$, and we define the matrices $S \in \mathbb{R}^{N\times N}$, $M \in \mathbb{R}^{N\times N}$ with

"Stiffness" matrix $\longrightarrow$ $\longleftarrow$ "Mass" Matrix

$$S_{ij} = \int p(x)\phi'_j(x)\phi'_i(x)\,dx \quad \& \quad M_{ij} = \int q(x)\phi_j(x)\phi_i(x)\,dx,$$

then our linear system has the form

$$S\vec{\alpha} + M\vec{\alpha} = \vec{z}$$

with
$$\vec{z} = \begin{bmatrix} \int g(x)\phi_1(x)\,dx \\ \int g(x)\phi_2(x)\,dx \\ \vdots \\ \int g(x)\phi_N(x)\,dx \end{bmatrix} - a\left(\int p\phi'_0\phi'_1\,dx + \int q\phi_0\phi_1\,dx\right)\vec{e}_1 - b\left(\int p\phi'_{N+1}\phi'_N\,dx + \int q\phi_{N+1}\phi_N\,dx\right)\vec{e}_{N+1}$$

OR

$$A\vec{\alpha} = \vec{z}$$

Observe:

① S + M are tridiagonal matrices. This structure is due to the choice of basis functions. If we choose higher-order basis fctns, then the bandwidth of the system matrix increases.

② As we make h smaller, we get a better approx to our solution. The restrictions on p + q guarantee invertibility of the matrix; ie. solvability of the system

③ Questions:

① Guarantee Convergence

② What does the weak form have to do with the original O.DE.?

③ We need to discuss numerical quadrature!!