From our previous discussion, we approximate $f \in H$ with

$$f = \sum_{i=1}^{\infty} \underbrace{\langle f, \phi_i \rangle}_{\alpha_i} \phi_i \quad \longleftrightarrow \text{ (Generalized) Fourier Series}$$

where

$\alpha_i = \langle f, \phi_i \rangle$ are the (Generalized) Fourier Coefficients

- The most familiar version of Fourier Series that you may have already encountered is the Trigonometric Series in Section 2.2.3

<u>Section 2.2.3 Trigonometric Series</u>

* Note → this discussion is taken from Kreyszig's book (begins on pg. 154 of his text)

Let $X = L^2[0, 2\pi]$ with the usual inner product

$$\langle x, y \rangle = \int_0^{2\pi} x(t) y(t)\, dt$$

- Define $u_n(t) = \cos(nt)$, $n = 0, 1, 2, \ldots$
  Then $\{u_n\}_{n=0}^{\infty}$ forms an orthogonal set in X.
  In particular,

$$\langle u_m, u_n \rangle = \int_0^{2\pi} \cos(mt)\cos(nt)\, dt = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n = 1, 2, 3, \ldots \\ 2\pi, & \text{if } m = n = 0 \end{cases}$$

So, $\|u_n\| = \sqrt{\langle u_n, u_n \rangle} = \sqrt{\pi}$ for $n = 1, 2, \ldots$
and $\|u_0\| = \sqrt{2\pi}$

} This will help us normalize the orthogonal set.

- In a similar fashion, define $v_n = \sin(nt)$, $n=1,2,3,\ldots$ then $\{v_n\}_{n=1}^{\infty}$ forms an orthogonal set in X.

$$\langle v_m, v_n \rangle = \int_0^{2\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m=n=1,2,\ldots \end{cases}$$

So, $\|v_n\| = \sqrt{\langle v_n, v_n \rangle} = \sqrt{\pi}$ for $n=1,2,\ldots$

- We can use each of these sets to build orthonormal sets:

Using $\{u_n\}$

Ⓐ Define $e_0(t) = \frac{1}{\sqrt{2\pi}}$, $e_n(t) = \frac{\cos(nt)}{\sqrt{\pi}}$, $n=1,2,3,\ldots$ $\left\{\frac{u_n(t)}{\|u_n\|}\right\}$

$\{e_n\}_{n=0}^{\infty}$ is an orthonormal set in $L^2[0,2\pi]$.

Using $\{v_n\}$

Ⓑ Define $\tilde{e}_n(t) = \frac{\sin(nt)}{\sqrt{\pi}}$ This is $\frac{v_n(t)}{\|v_n(t)\|}$

for $n=1,2,3,\ldots$

$\{\tilde{e}_n\}_{n=1}^{\infty}$ is also an orthonormal set in $L^2[0,2\pi]$.

- Note that we also have that $\langle u_m, v_n \rangle = 0$ for all m,n

i.e. $\int_0^{2\pi} \cos(mt)\sin(nt)\,dt = 0 \quad \forall\, m,n$

- Hence, we can build an orthonormal set $\{\phi_i\}_{i=1}^{\infty}$ using

(i) $\{e_n\}_{n=0}^{\infty}$ → Fourier Cosine Series

OR (ii) $\{\tilde{e}_n\}_{n=1}^{\infty}$ → Fourier Sine Series

OR (iii) $\{e_n\}_{n=0}^{\infty} \cup \{\tilde{e}_n\}_{n=1}^{\infty}$ → Full-Blown Fourier Series.

- From our discussion in the previous section, we know that if we want to use any of the previous orthonormal sets to approximate a function $f(t) \in L^2[0, 2\pi]$, then the expansion looks like

$$f(t) = \sum_{i=1}^{\infty} \alpha_i \phi_i(t) \quad \text{with} \quad \alpha_i = \langle f, \phi_i \rangle$$

- If we try to approximate $f(t)$ with the full-blown Fourier Series, then we write

$$f(t) = \underbrace{\langle f, e_0 \rangle e_0}_{①} + \sum_{k=1}^{\infty} \left[\underbrace{\langle f, e_k \rangle e_k}_{②} + \underbrace{\langle f, \tilde{e}_k \rangle \tilde{e}_k}_{③} \right]$$

Term①: $\langle f, \frac{1}{\sqrt{2\pi}} \rangle \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi} \langle f, 1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt$

Term②: $\langle f, e_k \rangle e_k = \langle f, \frac{1}{\sqrt{\pi}} \cos(kt) \rangle \frac{1}{\sqrt{\pi}} \cos(kt)$

$$= \underbrace{\frac{1}{\pi} \langle f, \cos(kt) \rangle}_{= a_k} \cos(kt)$$

Term③: $\langle f, \tilde{e}_k \rangle \tilde{e}_k = \langle f, \frac{1}{\sqrt{\pi}} \sin(kt) \rangle \frac{1}{\sqrt{\pi}} \sin(kt)$

$$= \underbrace{\frac{1}{\pi} \langle f, \sin(kt) \rangle}_{= b_k} \sin(kt)$$

So, we can simplify this a bit & rewrite in the more standard form

$\longrightarrow$ next page

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt)$$

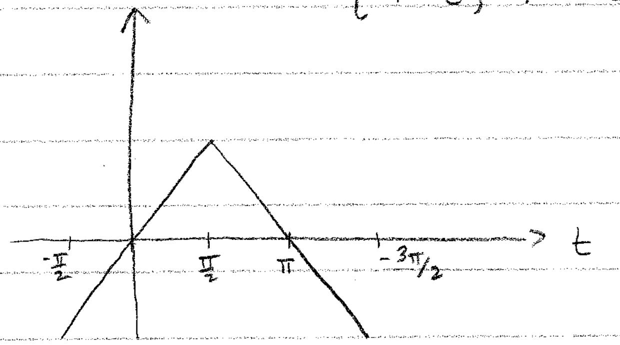
where

$$(*) \quad \begin{cases} a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t)\,dt \\ a_k = \frac{1}{\pi} \int_0^{2\pi} f(t)\cos(kt)\,dt, \quad k = 1, 2, 3, \ldots \\ b_k = \frac{1}{\pi} \int_0^{2\pi} f(t)\sin(kt)\,dt, \quad k = 1, 2, 3, \ldots \end{cases}$$

→ My "Fourier Coefficients" for $f(t)$.

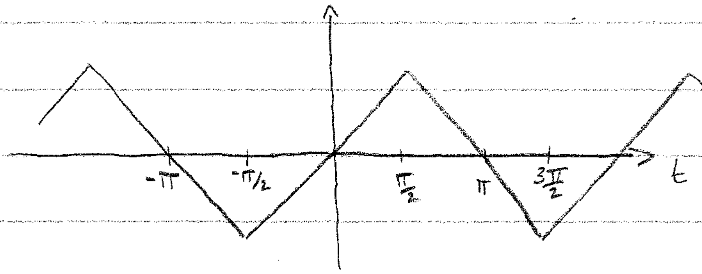
- Remark: We chose the interval $[0, 2\pi]$ because it's an interval of length 2π (so that $\sin(kt)$ and $\cos(kt)$ cycle through an integer # of periods over that interval. We can just as easily use any other interval of length 2π.

Ex: Let $f(t) = \begin{cases} t, & -\pi/2 \le t \le \pi/2 \\ \pi - t, & \pi/2 \le t \le 3\pi/2 \end{cases}$



Note: We consider this function on $[-\pi/2, 3\pi/2]$
And we could extend it to a periodic function on $\mathbb{R}$ by requiring $f(t+2\pi) = f(t) \quad \forall\, t \in \mathbb{R}$.

Picture of the periodic function looks like



Approximating this function with a Fourier Series, we see that $a_k = 0$ for $k = 0, 1, 2, 3, \ldots$ (Since $f(t)$ is an odd function, then $f(t)\cos(kt)$ is an odd function on $[-\pi/2, 3\pi/2]$, so the integral of the product is zero) - you may have to spend some time convincing yourself of this one! And

$$b_k = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(t)\sin(kt)\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} t\sin(kt)\,dt + \int_{\pi/2}^{3\pi/2}(\pi - t)\sin(kt)\,dt\right]$$

I.B.P. exercise!

$$= \frac{1}{\pi}\left[-\frac{1}{k}t\cos(kt)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{k}\cos(kt)\,dt\right]$$

$$+ \frac{1}{\pi}\left[\frac{-(\pi - t)}{k}\cos(kt)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{1}{k}\cos(kt)\,dt\right]$$

$$= -\frac{1}{k\pi}[0] + \frac{1}{\pi k^2}\sin(kt)\Big|_{-\pi/2}^{\pi/2} + -\frac{1}{k\pi}[0] + \frac{1}{k^2\pi}\sin(kt)\Big|_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi k^2}\sin(k\pi/2) - \frac{1}{\pi k^2}\sin(-k\pi/2) + \frac{1}{\pi k^2}\sin(k\pi/2) - \frac{1}{\pi k^2}\underbrace{\sin\left(\frac{3k\pi}{2}\right)}_{-\sin\left(\frac{k\pi}{2}\right)}$$

$$= \frac{4}{\pi k^2}\sin\left(\frac{k\pi}{2}\right)$$

So, $b_k = \frac{4}{\pi k^2}\sin\left(\frac{k\pi}{2}\right)$ for $k=1,2,3,\ldots$ so $b_k=0$ for k even

Then the Fourier Series approximation for $f(t)$ is

$$f(t) = \frac{4}{\pi}\left(\sin t - \frac{1}{9}\sin(3t) + \frac{1}{25}\sin(5t) - \frac{1}{49}\sin(7t) + \cdots\right)$$

- At this point in the discussion, I plan to drag out the laptop & show some Fourier series approximations that I've found online. (if technical difficulties are not prohibitive)

- Similar fourier coefficient formulas are available in your text for a general interval of length L, see eqn. (2.10) pg. 75

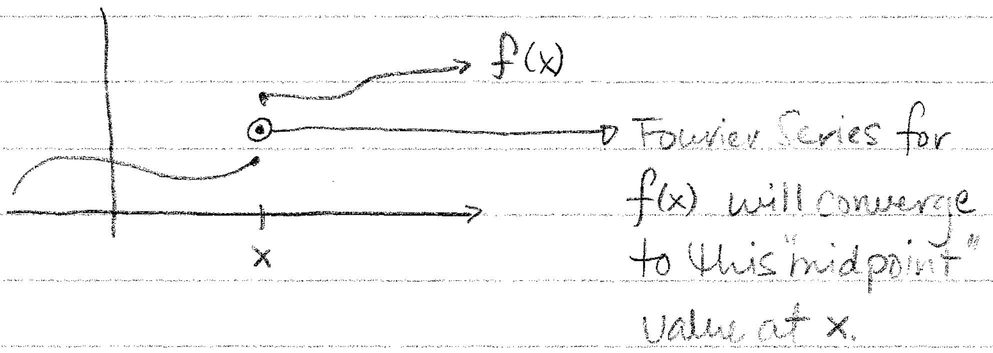
- Fourier Series "converge" in the L^2-sense, so we have to be careful when relying on it for pointwise convergence.

<u>Theorem 2.4</u> (Fourier Convergence theorem)
Spse $f(x)$ is piecewise $C^1[0,2\pi]$ (i.e. is continuous and has a continuous 1^{st} derivative except possibly at a finite # of pts. at which there is a jump discontinuity, at which left- & right-hand derivatives exist). Then the Fourier series $(*)$ of $f(x)$ converges as $n\to\infty$ to $\frac{1}{2}[f(x^+)+f(x^-)]$ for every $x\in(0,2\pi)$. At $x=0$ & $x=2\pi$, the Fourier series $(*)$ converges to $\frac{1}{2}[f(0^+)+f(2\pi^-)]$.

- This theorem says that at pts. x where f(x) is continuous, then the Fourier series in (*) converges to the value.
$$\tfrac{1}{2}[f(x^+)+f(x^-)] = \tfrac{1}{2}[2f(x)] = f(x)$$

- And at a point x where f has a jump discontinuity, then the Fourier Series approximation in (*) will converge to the average value of the left + right-hand limits, ie. to the value
$$\tfrac{1}{2}[f(x^+)+f(x^-)] = \text{midpt. of jump-discontinuity}$$

Picture



- At this point, I hope I have already pointed this out with some of the online examples. If not, you can Google "Fourier Series" and find some good pictures that illustrate this.

- Your text also points out that if f is continuous & not piecewise C^1, then f(x) and its Fourier Series may differ at an infinite # of points (ie. on a set of measure zero) - so the Fourier Series will converge in the L^2-sense.... but it still differs from f(x) at an infinite # of x-values. BE AWARE!!